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Mathematics in general education
H. C. TRIMBLE

A mathematics record card
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Mathematics in general education

H. C. TRIMBLE, *Iowa State Teachers College, Cedar Falls, Iowa.*

General education is herein presented as an avenue which leads to liberal or special education.

I SHALL DIVIDE my remarks into two categories. First I shall speak of the needs of society for more and better mathematics for general education. I shall limit the phrase *general* education in contrast to the phrases *liberal* education on the one hand and *special* education on the other hand. Then I shall speak of the needs of our pupils for mathematics for general education. I shall comment upon the problem of individual differences as it affects the potential learner.

It seems evident to me that the new industrial revolution will increase still further the demand for symbol-minded people. In an age of automation the demand for skilled labor increases. But the skills that people need to repair and program the machines are based upon ability to acquire and use new symbols.

The cookbook of the future will be much more complex than the cookbook of the past. The new cookbook will call for translating the symbols of mathematics into punches on a card, open or closed relays, and zigzag lines on a roll of graph paper.

More people will be needed to write the cookbooks of the future. These are the people who apply principles of science to devise new recipes for the cooks to follow. They must be well grounded in science; they must think in terms of isomorphisms between mathematical processes as integration and physical processes as summing the elements of work done by a variable electrical force; they must be skillful in explaining to the cooks the steps that produce the cake. To continue our metaphor, society will need more and better chefs.

Finally, society will need more epicures. Someone must savor the almost limitless set of permutations of ingredients and come up with promising new combinations. These are the pure scientists—call them artists if you will—who find form and logical structure where most of us find only raw data.

Now, what of the education of these citizens of the future? We have heard many claims for a liberal education that "frees the minds of men." I believe that the potential epicures will surely need such liberal education. The chefs will need a liberal education too, so will the cooks, if they are to contribute maximally in a free society. I believe that liberal education should take different forms for different people. Although its purpose is to free the minds of men, I believe liberal education should be grounded in religious and ethical traditions; that is, real freedom operates within a framework of values that society has learned the hard way.

I believe that general education is different from liberal education. Probably the three adjectives *liberal, general,* and *special,* as applied to education, symbolize a spectrum. It may be impossible to say precisely where liberal education leaves off and general education begins. Yet the mid-points of the two bands of the spectrum are clearly distinguishable. Liberal education should strive to promote individuality within a slowly-evolving framework of ethical and moral values.

General education should strive to make the individual a well-informed member of his social group. This means keeping lines of communication open. The individual should view the role he plays in relation to the roles that others play. His education should continue throughout his lifetime as he uses much of his leisure time to make his contribution to his social group.

This concept of general education leads many amateur philosophers to view the task of general education as primarily a job of instilling social attitudes. They concentrate upon making the pupil a *member* of a social group. The approach to the task often becomes a head-on attack upon the pupil's attitudes. Course content becomes a secondary consideration. Subject matter becomes at best a vehicle for building social values.

To conceive general education primarily in terms of attitudes misses, I think, an important point. Children acquire language skills faster than adults do. It is all very well to have good social attitudes. To be a well-informed member of a social group takes more than this. Language skills are needed; in particular, the citizen of the future will need more and more of the language of mathematics.

I think that those educators who claim that what you study is less important than how you study it, have a good point. Many scientists, including Professor Conant, have recommended that an inductive study of a few problems of science should replace the familiar, deductively arranged, survey courses in science for general education. There is a great deal of evidence that in content fields like literature, social science, and natural science, one set of topics may serve the purposes of general education as well as another set of topics.

The point where this carefree choice of subject matter breaks down, it seems to me, is the point where you begin discussing language skills. Of course you can learn something about the English language by discussing home management. I believe there are other things about English that you can learn only by a conscious, organized study of English as such.

In our own area of mathematics we went through a period of teaching junior high school pupils what we called functional mathematics. In the 1930s this meant mathematics that would operate (or function) in everyday life. We taught topics like budgeting and insurance, with a little number work dragged in by the heels. Unfortunately, many textbooks in current use continue this tradition.

When you try to teach mathematics in this way, you find that it doesn't work. It is easy to understand insurance when you understand ratios; but pupils do not learn much about ratios as they study insurance. At best, students learn to figure premiums from the data the textbook problems present. But when you change the topic (or even the wording of the problems) ever so slightly, you must begin teaching the pupils "how to do it" all over again.

The newer organization of junior high school mathematics is around "key concepts and underlying principles" of mathematics, concepts like place value, comparison by subtraction and by ratio, and the like. Of course you approach these concepts through problems; but you choose problems much more interesting to the children than the problems in budgeting, taxation, and insurance that the older textbooks featured. Once the basic mathematical ideas have been taught, it is easy to apply them to these more adult topics.

I do not deny that there is content in mathematics for general education. I believe there are things everyone should know, but I believe these things are best thought of as concepts and that language adapted to expressing these concepts is required. If you say everyone must be able to compute his income tax, I believe you miss the point. You should say, rather, everyone should have the concepts and symbols that he needs to follow the directions for computing his income tax.

When you apply this principle to organizing a college program of mathematics for general education, you get some startling results. Some people have said everyone needs trigonometry; then they felt obliged to teach trigonometry as you would teach it to a future surveyor. We should say, rather, every college student should be able to read a lucid set of directions for solving an oblique triangle. You see at once that he will need to have contact with the trigonometric functions, so you help him to extend his concept of ratio to include trigonometric ratios. Then you guide him through a few typical applications of trigonometric ratios.

Some pupils will come to life in such a program of mathematics for general education. They will find a starting point for the kind of liberal education that fits them, or they will find a line they want to follow as a specialist, whether as a vocation or as a hobby. Most pupils will merely learn the communication skills that they need to keep in touch with their more mathematically-minded colleagues.

So the purposes of mathematics for general education, as I see them, are:

a) to give the learner leads that he may or may not follow toward further liberal or special education;

b) to give the learner the language skills that he needs to be a well-informed member of his social group.

I pass now to the kind of program of mathematics for general education that I think is most likely to achieve these purposes.

Some of my friends agree with the purposes I have stated and with the rationale that I have used as a build-up for these purposes. Then they say, "Let's go to work to build the language skills that a well-informed citizen needs." They come up with a basic skills type of course. They define the symbols of mathematics carefully. Then they provide lots of drill materials to give the pupils practice in applying these symbols.

My objection to basic skills courses in college mathematics is based upon my experience that they do not work. The most able pupils find them boring. The less able pupils find them an even less comprehensible repetition of things they didn't understand before.

Others of my friends avoid the basic skills course for somewhat the reasons I have stated. They look for a fresh approach through topics in modern mathematics, tracing the history of a mathematical idea, or the like. I like this approach. It can be stimulating for both the teacher and the pupils. It is much more likely to provide leads toward further liberal or special education than is a basic skills course. If the teacher does a good job, the pupils will acquire some of the language skills they need; they will develop attitudes conducive to continuing to learn after they have completed the course.

In practice, these fresh approaches to mathematics for general education have often run into snags. The most common snag is a result of trying to teach too little about too much. The pupil ends up with a great deal of undigested information. A second snag is a result of trying to teach running to pupils who have not yet learned to walk. Abstractions are made by people who have something to abstract from. I have heard pupils use the term *abstrocity* to describe what were, to me, beautiful generalizations. A third snag is the result of following too far an idea that interests the teacher or the textbook writer. The pupils easily lose sight of the trees as they examine the forests.

I believe that a good program of mathematics for general education will deal with problems that are new to the pupils. The heart of the course will be a reaching out for language that is appropriate to an adequate formulation and solution of a problem. The pupil should be made a partner in this search for language. Only in this way can he acquire an appreciation for the elegance of the language of mathematics.

Once a pupil begins to enjoy finding a

language in the terms of which he can express ideas, teaching becomes easier. You can teach this pupil more of the language of mathematics per unit of available time, and you should try to select from the vast store of such language the portions most likely to be useful to a well-informed citizen.

This brings me to my final task of commenting upon the problem of individual differences. A teacher of mathematics for general education must accept the fact of individual differences and plan accordingly.

In the realm of special education, administrators tend to assign teachers to their areas of special competence. Then it becomes easy for the teacher to conceive his job. With rather good justification, he assumes that his pupils know nothing initially. He defines a successful pupil as one who finishes the course knowing most, but not quite all, of what he himself knows.

This way of work doesn't fit the demands of general education. Pupils come to a college course in mathematics for general education with widely different backgrounds. As a result, they differ in what they know, what they believe is worth knowing, how they try to learn, and, to be brutally frank, what they are capable of learning. In general education it is surely true that "one man's food is another man's poison."

Under these conditions it seems to me that the simple, absolute standards of special education become ridiculous. To preserve his sanity, if for no other reason, the teacher of general education must learn to roll with the punch. He does what he can with the pupils he gets. This is a serious and difficult assignment. It means withholding *A* grades from students who knew a great deal initially and didn't learn much more. It means providing things that these advanced students may reasonably be expected to learn to earn their *A* grades. At the same time it means pacing class activities to give retarded pupils the feeling of security that they need.

So the problems upon which the good program of general education are based must be broad enough to provide useful activities for pupils at all of the levels of development represented in the class. Within these problems the teacher must clearly define a minimum core of information and skill that he expects every pupil to acquire. Within these problems the teacher must find leads that go beyond the core in different directions, and to different depths, to interest and challenge individual pupils.

It seems obvious that each college will need to pitch its mathematics for general education at a level appropriate to its own student body. Even when this has been done, there will be wide differences from pupil to pupil in what they bring to the course and in what they take from the course.

Despite these limitations, the purposes of mathematics for general education are invariant from college to college, and from pupil to pupil within a class. Each pupil should be expected to acquire as much as he can of the concepts and language of mathematics that a well-informed citizen needs. Each teacher should be expected to help individual pupils find leads toward further liberal and/or general education.

A mathematics record card

CLARENCE OLANDER, *St. Louis Park High School, St. Louis Park, Minnesota.*

An adequate system for keeping records forms the basis of good counseling programs, research studies, and curriculum studies. This paper describes the efforts of one group of teachers to develop a good system for keeping a record of the students' achievements.

IN MATHEMATICS, as well as in other areas of the curriculum, students differ in many ways. There are differences in mental ability, achievement, rate of learning, apparent ability, readiness for new learning, interests, attitudes, and study habits. These differences exist for a variety of reasons, among which are such personal factors somewhat beyond the influence of the school as home environment, and physical and mental health. Recognizing the existence of these individual differences is of itself inadequate. The schools must be prepared to answer such questions as the following: What are ways of providing for individual differences? How shall the slow pupil acquire new skills? What techniques and activities are most effective in challenging the more capable students?

A quick review of the literature will reveal methods that have been effective in providing for individual differences. Supervised study, differentiated assignments, group organization and group instruction within the class, individualized instruction using self-instructional material, enrichment activities for the capable student, homogeneous grouping, remedial classes, the project method, and the provision of a dual or multiple curriculum are methods that have been used more or less successfully at one time or another. However, regardless of the methods used, the basic premise still remains: the teacher must know the pupil.

It is the purpose of this paper to describe a mathematics record card that is currently being used by the mathematics department of the St. Louis Park High School. This record card provides each teacher at the beginning of the year with information necessary to determine the status of each pupil in his class with respect to achievement, ability, needs, and several other factors. This record card, as reproduced here, will show the record of a student throughout his mathematics career in grades seven to twelve. All cards are filed in a central location at the end of the school year to be available for each teacher to pick up cards for all his mathematics students at the beginning of each year. Seventh grade teachers have the responsibility of preparing a new card for each student entering seventh grade. A card must be prepared for a student coming into the system at any level. After graduation, or after a student has stopped taking mathematics courses, his card is placed in the student's cumulative record folder in the counselor's office.

Each teacher adds relevant information to the card during the course of the school year. For example, if an intelligence test is administered by the counseling depart-

ment in grades seven and ten, the teachers in these grades look up the results and record them on the cards.

Some of the information presented on the card is of an objective nature, such as test scores, while other information is more subjective and requires teacher judgment. Blank spaces are provided so that teachers may make note of information not specifically requested on the record card itself. Thus, as the student progresses through the junior and senior high schools, more and more information is added. By the time the student reaches the upper grades, the teacher has a rather complete and comprehensive record of each student. This is invaluable in providing for individual differences as well as in providing data useful for several other purposes. In compiling a record such as this, each teacher shares in the busy work and thereby obtains maximum information with a minimum of effort. Furthermore, the subjective judgment of several teachers appears on each card. This is desirable since it provides a consensus of opinion regarding each student.

The test scores recorded on the cards are the result of well-designed testing programs. The counseling department conducts a comprehensive testing program throughout the year. Those scores, such as scholastic ability, which are considered most appropriate and useful for mathematics teachers are recorded on the cards by the teachers at the level at which the test is administered. The results of many other tests are available in the counselor's office. However in order to keep the mathematics record card as simple as possible, and still have it remain useful, all scores available in the counselor's office are not recorded.

In addition to the testing program of the counseling department, the mathematics department conducts a testing pro-

ST. LOUIS PARK HIGH SCHOOL MATHEMATICS RECORD CARD

NAME _____ (Last) (First) (Middle)

Grade and Course		Elementary School Data	7	8	9	10	11	12
Basic Skills	Computation							
	Reasoning							
Math. Coop. (P. R.)								
Algebra								
Geometry								
Scholastic Ability	I. Q.							
	A. C. E. (P. R.)							
Apparent Ability (Letter Grade)								
Final Achievement (Letter Grade)								
Difficulties								
Strengths								
Vocational Plans								
Projects								
Name of Teacher and Date								

gram of its own. Currently this program is in the early stages of its development. The present program consists of an arithmetic skills test and a functional competence test. The basic skills test is given at the beginning of each school year to all students in grades seven to nine. The functional competence test is given to all ninth-grade students at the beginning of the year. Consequently the results of these tests are available to the teachers early in the school year. At St. Louis Park High School, the testing program is supplemented by standard tests given by individual teachers. Most teachers in the junior high school give a basic skills test at the end of the year. Most of the algebra and geometry teachers use a standard test in these areas as a final achievement test. Many of the students in the senior high school mathematics courses are given a basic skills test. In a few instances, an algebra test is repeated a year after the students have completed elementary algebra to measure retention. Some students are given an algebra aptitude test in the eighth grade. This test will probably be given to all eighth-grade students within the near future as part of the testing program.

It is not our purpose to discuss the testing program as such, but rather to give the reader an idea of some of the different kinds of test scores that are recorded on the card. Obviously, the information available will not be the same for all students. Some teachers will obtain more scores and information than others. However, certain data will be available on all students. This will increase as the testing program becomes more comprehensive.

To obtain maximum benefits from this record card, the test results must be recorded in a form which will be meaningful to each teacher. Local percentile norms are being prepared for the basic skills test and functional competence test. Standard scores are available for the achievement tests. The teacher simply records a numerical score on the card. This may be a raw score, standard score, or a percentile rank. A supplementary sheet is prepared each year to accompany the mathematics record cards. This sheet lists all the tests that have been administered along with pertinent standardization data. This includes percentile norms, means, medians, standard deviations, standard scores, and standardization populations. Eventually local norms will be prepared for most tests that are used, thereby simplifying the entire process of recording and interpreting the scores.

The test scores that are included on the record card enable teachers and administrators to group students in such a manner that individual differences can be provided for effectively. Other objectives can also be realized through a testing program and record card system of this type. These objectives include:

1. To provide data concerning the retention of certain facts, skills, and abilities in the areas of arithmetic, algebra, and geometry.
2. To provide data through which growth in certain skills and competencies may be measured.
3. To determine the influence of certain courses such as algebra and geometry on the retention of certain basic skills of arithmetic.
4. To determine the influence of geometry on the learning and retention of algebra and the influence of advanced algebra on the retention of geometry.
5. To determine the influence of general mathematics courses at different levels on the development and retention of topics included in the areas tested.
6. To provide data through which the achievement of students in the traditional mathematics sequence can be compared with students in various general mathematics courses in comparable areas such as basic skills.
7. To provide data whereby the effectiveness of different methods of curriculum organization can be compared.

8. To determine the level of difficulty of certain aspects of the high school mathematics program.
9. To provide students, teachers, and counselors with information about students to be considered in educational and vocational planning.
10. To assist schools to identify early in a student's academic career anyone who has talent for further academic work.
11. To provide to educators a continuous inventory of the achievement of students so that criticisms directed at public education may be answered with objective evidence.
12. To provide data for the study of the science of predicting success in mathematics.

A study of these objectives will make apparent the fact that data must be obtained and recorded for a period of several years before significant results can be obtained. However, certain interesting comparisons can be made on the basis of data obtained after a period of one or more years.

In addition to the various test scores discussed above, other information is also reported on the record card. Each teacher makes an estimate of the student's apparent ability. This estimate is based on aptitude and achievement test scores, achievement in previous grades, and teacher observation of the student involved. Final achievement grades (marks) are recorded for each student. By comparing apparent ability and actual achievement, the teacher can reach a conclusion concerning how well a student is achieving compared with his ability to achieve.

Provision is made on the card for recording strengths and difficulties of each student. These categories do not have to be filled out for all students. If a student is extremely weak in one area, such as division of fractions, this fact should be recorded. Appropriate steps involving remedial teaching can then be taken by his present or subsequent teacher to clear up the difficulty. A sound program of diagnostic testing will be useful in discovering these specific weaknesses.

The information on vocational plans will help a teacher determine if the student's interests are consistent with his abilities. Course work in some mathematics classes can often be adapted to the vocational plans of the students. A record of the projects that a student has completed can be useful in arranging for further individual activities on the part of the student. This will also be useful in determining specific interest of students.

A space on the record card is provided for the name of each teacher and the year the student takes a course. Since several different courses are available in grades nine to twelve, the name of the course is listed after the grade. The vertical lines in the spaces following "Basic Skills" divide these spaces so that scores can be recorded for tests administered both at the beginning and the end of the year. The card does not supply any personal data other than the name of the student. Complete personal records of the students are available in the counselor's office.

This record card is useful in assisting teachers to prepare forms and letters of recommendation for their former students. Colleges and prospective employers request material of this kind but teachers often lack relevant data. The record card system described in the above paragraphs insures the availability of such data.

The use of this record card has focused the attention of the mathematics faculty on the desirability of a sound testing program. It has resulted in renewed attention on the problem of individual differences. The work involved in the compilation of the record card is more than compensated for by the benefits derived from such a card system. In the future, the card may be modified to implement changes suggested by the teachers. Possibly suggestions from the reader would make the card more satisfactory!

Mathematical science and the manpower problem

RAYMOND J. SEEGER, *National Science Foundation, Washington, D. C.*

A scientist asserts that " . . . the problem of teaching mathematical science is primarily a problem of understanding mathematical science."

EVERYONE IS AGREED that there is a serious manpower problem with respect to science. As you know, I am not an expert on manpower. Being a physicist, however, I do have two personal convictions in this regard. First, the manpower problem with respect to science is essentially a problem of teaching science, particularly, mathematical science; and secondly, the problem of teaching mathematical science is primarily a problem of understanding mathematical science.

Modern times are popularly called the atomic age—to be sure, they cannot properly be designated a scientific age. There is, indeed, a widespread failure to differentiate between science and technology, between basic research discoveries, such as the identification of nuclear energy, and later applications, such as atoms for bombs and atoms for peace. Associated with this translation of scientific information into technological use, there has been a continually increasing demand for specialists, both scientists and engineers, and, in turn, an increased use of such specialists. For example, while the U. S. population[1] has doubled since 1900 and the number of professional specialists has quadrupled, the demand for the number of engineers has increased five times as fast as the population and the demand for scientists ten times. Increased educational programs, moreover, have not been adequate for satisfying this accelerated demand. It is true that the number of college graduates (as a per cent of the population aged 22) increased from seven to eight times between 1900 and 1950, but the corresponding number in the humanities has a little more than trebled, and that in the natural sciences has increased only about five times.

Although a comparison between U. S. professional personnel and those in the U.S.S.R. should never be regarded as a guide line for the development of our own country, from a national point of view pertinent figures[2] cannot be ignored. In this instance, they turn out to be not at all encouraging. In 1953–54, although the Soviet Union had only about 60% as many graduates as the United States, 60% of them were in engineering and the sciences, whereas less than 25% of the graduates in the United States were so classified. With regard to total professional graduates, the Soviet Union had about 80% more agriculturists than the United States, about twice as many engineers, and about three times as many physicians.

All in all, as far as engineering and the sciences are concerned, there has been a steadily growing shortage of qualified specialists in our country.

One would naturally turn to the source of specialists for a remedy, namely, to the

[1] All U.S.A. statistics here are from D. Wolfle, *America's Resources of Specialized Talent* (1954).

[2] All U.S.S.R. statistics here are from N. DeWitt, *Soviet Professional Manpower* (1955).

teachers. Here, too, unfortunately there is a continuing shortage. An annual deficiency of more than 60,000 teachers is now estimated for elementary and secondary schools. By 1970 the college student population is expected to be twice what it was in 1950. This pleasant prospect, however, is offset by a serious faculty shortage that will probably develop in the colleges as early as 1960, a condition that has been aggravated in recent times by the low-salary scale and by the low-prestige value associated with the teaching profession. Here, too, a comparison between the Soviet Union and the United States is discouraging. Although the number of education graduates in mathematics and science in the Soviet Union is only about one half of those in the United States, the majority of these graduates in the Soviet Union actually become science teachers, whereas in the United States only about half of them do so. Moreover at present, many such U. S. teachers are being enticed into industrial and government employment. Thus there is a steadily growing shortage of qualified teachers, a shortage at the very source of the supply of specialists—hence my personal conviction that the manpower problem with respect to science is essentially a problem of teaching. Let us, therefore, consider now the teaching situation itself.

Dr. James B. Conant has urged that "to find and educate the gifted youth is essential for the welfare of the country. We cannot afford to leave undeveloped the greatest resource in the country." In this connection, I am reminded of Alan Greg's cow test for identifying gifted youth. You say to a child, "Suppose you call the tail of a [four-legged] cow a leg, how many legs will the cow then have?" If the child replies immediately, "Five," encourage him to take mathematics. If he observes that it doesn't make any difference what you call a tail, "It is still a tail," suggest science as a career. If he hems and haws and finally comes up with an answer like seven or eight, plead with him not to go into science or engineering. If, however, with a twinkle in his eye he says that the question is a good one, mentally note that he will become an administrator.

Let us turn our attention first of all to the question of ability, specifically with respect to identification and encouragement. It is a well-known fact that intellectual accomplishment is generally associated with high intelligence—not surprising in view of the fact that the words intellect and intelligence both come from the same root words *inter legere*, meaning to choose between, to discriminate. Outstanding scientific achievement, too, is definitely associated with high intelligence. The primary problem, therefore, is one of quality rather than one of quantity; not so important how many, as who. I know of no specific number of second-class mathematicians that can be regarded as qualitatively equivalent to one first-class mathematician. Evidently there is a primary need for identifying gifted children as early as possible.

Once the identification has been made, the next step is appropriate encouragement. It is distressing to realize that in this country less than half of the high school graduates who are potential college graduates even enter college. Less than one-quarter of high school graduates ever become college graduates. Less than half of the high school graduates in the upper quarter of their class become college graduates; only 60%, indeed, of high school graduates in the top 5% of their class graduate from college. If more highly intelligent people are to be secured in the professions, first of all more of them must be encouraged to go to college. It is important to keep in mind that equal opportunity does not necessarily signify identical opportunity. In this case, indeed, opportunity should preferably be in proportion to ability. How, then, does one identify ability?

I must confess certain deeply rooted prejudices about common practices in

this regard. In so much of our training we are concerned with seemingly irrelevant questions: for example, "Who is your father?" "Is he not a carpenter?" When I was a graduate student, I heard a professor of eugenics speak on the subject, "Who Shall Inherit the Earth?" His answer included the children of the generals and the admirals, of the statesmen and the businessmen, of the doctors and the lawyers, of the preachers and the professors, *et al.* When he finished, there rose an old man—a newspaper editor and former ambassador to China—who said, "Mr. Professor, you are right; those are the people who will inherit the earth. I will now tell you about the people who will take it away from them: the sons and daughters of the Irish charwomen on State Street."

From time to time I have jokingly inquired of my astronomy friends as to potential astronomers among American Indians (cf. the Cherokee Sequoia) in view of their evironmental and hereditary interest in the skies. Last spring Dr. Warren Weaver told a story about a Mexican Indian which well illustrates my concern. A secondary school teacher of music was visiting a small village about four hundred miles from Mexico City. Noting an attractive Indian boy of ten, he inquired of the boy if he would be interested in assisting at a musical performance. Something of the boy's enthusiasm stirred the music teacher deeply. The next day he sought the parents' permission to let the boy become his personal valet. The parents who had already seventeen children and were expecting still another were quite happy with this prospect for the boy's financial aid and his social advancement. Within a couple of years the music teacher discovered that the boy had definite musical talent, which he proceeded to develop. At the age of nineteen the young man participated in a musical concert attended by President Porfirio Diaz, who awarded him a scholarship for study abroad. The Paris Conservatory of Music, however, regarded the young man as beyond the training age. Accordingly, he had to learn by himself. He became a concert violinist, director of a symphony orchestra, dean of a school of music, an investigator of bird music. His son graduated with a Ph.D. in science from Harvard and is today the president of a large university. You can readily understand my prejudice, therefore, when I hear people inquiring doubtfully, "Who is your father?" How well do any of us average people know and understand our own latent potentialities or those of others—not to mention those of children, including our own?

Another statement that stirs me somehow the wrong way is the casual remark, "Oh, you are a girl!" When Muriel Jane was in the second grade, she came home to inform me joyfully that her class was looking at rocks. "Ah, ha," I said, "you are now studying science." "Oh, no," she said sadly, "Only the boys are studying the rocks: the girls must practice dancing."

One wonders what master mind is assigning certain professions to women and others to men—or what chance schedule is making such distinction inevitable. Granted that for personal or for social reasons all women might not wish—actually do not wish—and perhaps, should not wish to become scientists and engineers, or even to go to college, should not each individual have an opportunity to do so if she so desires? Certainly doors of opportunity should be available—and open. Only 43% of the high school girl graduates that could potentially go to college do so, whereas in the case of boys, the corresponding figure is 65%. Yet the percentage of students graduating from college from among those beginning college is the same for both men and women. In the case of mathematics, twice as many men as women obtain the bachelor's degree. For graduate work in mathematics, however, there are more than four times as many men as women completing

the master's degree, and about twenty times as many men as women attaining the doctor's degree. Why? One wonders if this is at all in accord with intellectual potential.

Again, not to confuse American culture with that of the Soviet Union, but for the sake of comparison, it is significant that in the Soviet Union women constitute more than one quarter of the engineering students, whereas in the United States they form less than half of one per cent. As a matter of fact, half the professionals in Russia are women.

Some time ago when I attended a U. S. conference on mathematics research, I inquired where the women were. I was informed by a distinguished mathematician that women could not qualify technically. At the time I wondered, and still do, if this condition results fundamentally from some innate difference between the sexes or merely from some imposed differentiation in early training and hence in later opportunities. After all, two Nobel prizes have been awarded only once to the same person, to Marie Curie, a woman. I believe we need to identify and encourage both women and men in accord with their intellectual abilities and with their personal interests. Let us now consider the training problem itself.

There are essentially two outlooks—a widening, horizontal one and a cumulative, vertical one. The broad horizontal outlook signifies man looking out upon his environment.

Some years ago in giving a course in physics, I invited a professor of history to talk to the class about the relation of history to science. He asked rhetorically, "What is history?" He himself answered, "The study of man and his environment!" I suppose everything could be thus classified vaguely as social studies. There is only one strictly human problem—man and his environment. Because of its complexity, however, we must approach this basic problem from different points of view.

When Muriel Jane was in the third grade, she brought home some drawings of planets and stars. "Ah, ha," I said, "you are now studying science." "Oh, no, Daddy," she replied, "This is social studies." Not wishing to appear to be an old fuddy-duddy daddy who is not alive to the modern age in which he lives, I said, "Of course." Later she brought home some material on dinosaurs. I cautiously inquired, "Is this work possibly science?" She smiled patiently as she said, "Of course not, Daddy, this is social studies." Later that year I had occasion to visit her class. One of the bright youngsters asked, "Teacher, isn't this material on dinosaurs really science?" The teacher replied, "I always thought so, but the book says it is social studies." Later I was describing my confusion to a fellow member of the Federal Schoolmen's Club. "Oh," he said, "I can explain your dilemma. In social studies we are interested in our neighbors —there is our neighbor, the moon; there are our neighbors, the dinosaurs." At last I saw the all-inclusive relationship, but neither the teacher nor the pupil apparently had ever understood this point of view. I believe we have to be very careful today in stressing relatedness so that we do not confuse it with identity.

Let me give you another illustration specifically involving mathematics. I once read a book on teaching methods for the junior high school. It contained some examples of typical class presentations, one of which dealt with geometry. The teacher in this instance was supposed to begin the discussion by pointing out that geometry has to do with laws. She would then ask the class what they knew about laws. By trial and error she would have to obtain the idea she wanted from one of the youngsters, namely, that the Constitution of the United States involves laws. Happily the teacher could then say, "Let us now study the Constitution in order to understand at first hand the concept of law." The entire period could then be devoted to an analysis of our constitutional law.

Unfortunately, what the teacher herself did not realize is that the concept of law as embodied in the Constitution is quite different from that found in geometry. Regardless of how long or how much one might study the Constitution, one would never learn geometry. In a real sense, may we not be betraying our children by depriving them thus of their intellectual heritage of mathematics? In trying to reduce everything to its social aspects we may actually be denying them an understanding of their total environment.

In this respect, a look at the curriculum of the Soviet Union is quite startling. Thirty-five per cent of the work there from grades one to ten is concerned with mathematics and the sciences (29% for grades one to four, 36% for grades five to seven, 41% for grades eight to ten). Of the science and mathematics taught in grades one to ten, two-thirds is mathematics and one-third science. Perhaps, such concentrated effort represents the opposite extreme from our U.S. laissez faire system of electives.

The fact is, however, that the world about us is not just a social environment. We can never understand this world purely in the terms of social conditions and of social relations. It is also a world of physical, biological, and psychological phenomena. Our children must be taught to respect and to regard the universe from these viewpoints. Furthermore, because of the central role played by mathematics in the interpretation of phenomena, it is imperative that at all educational levels the student should see mathematics in its relationship to physics, to biology, and to psychology, as well as to sociology. I would say, therefore, that our horizontal glance at man and his environment requires a broader outlook in the future, and that the present predominant focusing on social aspects is too narrow.

There is also a vertical outlook penetrating each horizontal level. Starting with the students in the graduate school, we find that in the natural sciences 83% (the highest percentage of all disciplines) are continuing their undergraduate major. A student, however, does not begin to major in mathematics or in science in his junior year in college without having taken some freshman work in these subjects. His election of such disciplines, accordingly, is predicated upon the interests and skills he has developed in secondary school. Evidently the selection at this time depends upon the interests and skills that have previously been developed in primary school. Hence I am persuaded that the whole problem of training in mathematical science is not peculiar to any particular grade, but rather extends generally through all grades, which are inherently interdependent. What the elementary teacher teaches inevitably bears on later choices by the student, and what the college teacher imparts depends obviously upon the student's earlier preparation. Teachers at the various levels need to get together and to discuss their common problems; they should not be content with discussion at their own horizontal level. We all need to take temporal views of educational training and to effect smoothly the transitions from one horizontal level to another. Thus in identifying and encouraging young people we need both a broader (horizontal) and a higher (vertical) view from the standpoint of ability.

It is rather interesting that mathematics was the Greek word for all learning—for the most advanced learning at that time was mathematics. When we look at the later medieval quadrivium, viz., arithmetic, geometry, music, and astronomy, we note that these, too, are all essentially mathematical. In the three Rs of yesterday we find one survival, arithmetic. In recent times, however, there has been a growing tendency at all levels of instruction to eliminate mathematics from general or liberal education. I suppose this movement has expanded owing to a widespread impression that nowadays mathe-

matics is poorly taught, possibly because of the intrinsic nature of the subject. Many people, therefore, have castigated mathematics teachers supposedly because of an overemphasis on problem solving, or because of an overspecialization of abstract concepts. Undoubtedly there is always a need for better teaching in all subjects. It would seem, however, that reformation of teaching is needed, rather than resignation to no teaching. Elimination of mathematics from the curriculum for all will certainly not improve the understanding of mathematics for any. Perhaps one element in this complex problem has been the fact that what has been taught in the name of mathematics has not really been mathematics at all, but rather certain numerical aspects of sociology.

Recently I read an amusing incident about a comic strip character called Susie Q. She was asked how she was doing in arithmetic. She replied that arithmetic had been easy until the class got to addition and subtraction. In her interesting book *Children Discover Arithmetic*, C. Stern notes about a six-year old, "She didn't like arithmetic because she got so tired of coloring rabbits." Granted that to a degree the teaching has been wanting, has it really been mathematics teaching?

Let us consider, therefore, the teaching of mathematical science. In this connection I would propose three axioms.

The first axiom for teaching mathematics, I would say cryptically, is to teach mathematics. Mathematics should be taught not as the dead accumulation of the past, but as a living challenge for the present. I appreciate the fact that a teacher should not teach a subject per se —without John Mark. On the other hand, neither can a teacher teach John Mark per se—without a subject. Actually, a teacher teaches mathematics to John Mark. Of course she must know John Mark. Likewise she must know mathematics. (Unfortunately this relationship is not sufficiently emphasized in colleges where the education—John Mark—department and the subject—mathematics—department usually teach teachers quite independently —and sometimes even antagonistically.)

Under no circumstances should mathematics be regarded as some undigested, or predigested lumps of "Information, please." It certainly should not be looked upon as a painless entertainment. Mathematics is, and must be taught as an independent and consistent subject involving a logical structure of its own. To be sure, it is essentially abstract and, therefore, necessarily difficult to communicate. There are two ways, I believe, to approach this problem.

In mathematics the approach should preferably be genetic, rather than systematic. In other words, at precollege levels one should stimulate the student with appreciation of the original, psychological, and sociological development of mathematical concepts, rather than with emphasis on comprehensive and detailed relation of logical deductions. Secondly, the approach should be more generally intuitive than specifically logical. Youngsters should be encouraged to look into mathematics itself, to seek revealing insights; certainly, in this sense mathematics will be more caught than taught. The primary goal will always be the understanding of mathematics—both its analytic and its synthetic phases, both its logical and its intuitive relations, both its general and its individual aspects.

I am reminded here of a parable of a tree, told by Felix Klein. When a tree grows, it does not develop one part completely, say, just the roots, and then later the trunk, and finally the branches with the leaves. On the contrary, it develops roots and trunk and leaves more or less simultaneously. The higher the trunk, always the longer the roots. So, too, in the growth of mathematical ideas, one must develop to some degree all aspects simultaneously at each educational level. In any case, therefore, let us begin mathematics teaching by teaching mathematics.

The second axiom, I would suggest, is

to teach mathematics with applications. Historically we begin here with a reciprocal relationship. Probably all mathematical developments have been rooted in practical problems. Conversely, scientific developments have certainly been furthered by purely mathematical formulations. The artificial separation of theory and of practice, originating with the Greeks and transmitted by them to the rest of the world, greatly retarded the development of all mathematical science. It is somewhat ironical that we practical moderns should hold in such high esteem Euclid of Alexandria, who never discussed how to draw either a straight line or a circle, who never gave a formula for determining the area of a triangle in terms of its base and its height, who never calculated π from the ratio of the circumference of a circle to its radius. I would not minimize Euclid's outstanding contribution in his comprehensive collection and axiomatic presentation of the geometrical knowledge of his day—a feat more properly understood at a graduate school level. It was Archimedes of Syracuse, however, who calculated π to be approximately 22/7, and it was Archimedes who used other mathematical methods than those of geometry to investigate physical phenomena. Archimedes was certainly a greater mathematician than Euclid. What is more, he exemplified the spirit and method of modern science, epitomized in the well-known Archimedes' principle.

The practical consequence of the Greek divorce between theory and practice is seen notably in the Greeks' failure to develop trigonometry, which had to await the observational requirements of terrestrial surveying and of astronomical mapping. More recently, too, those delights of modern mathematicians, infinite functionals and Hilbert space, were not invented in an ivory tower. On the contrary, they originated in the attempt of Jean B. J. Fourier to solve a practical problem of heat conduction. Thus we perceive the need to apply pure mathematics with its independent and consistent nature to the interpretation of phenomena. Because of the ease of experimentation with respect to physical environment, the physical sciences exhibit historically the outstanding example of the value of this interplay with mathematics. Strangely enough, creativity in mathematics is enhanced by contacts outside mathematics itself.

The third axiom I would urge is to teach mathematics with applications from a modern point of view. It is generally agreed that modern mathematics began in the seventeenth century with Descartes' conception of analytic geometry in 1637 and with Newton's invention of calculus in 1671. Yet many high school courses in mathematics are regarded as being quite satisfactory if the students know merely the binomial theorem in algebra, which was discovered by Newton in 1676, and have studied the first four books of Euclid, which were compiled about 325 B.C. The fanatic adherence to Euclid today is beyond belief. To be sure, we are greatly indebted to Euclid for his rigorous use of logical development and for his careful exemplification of deductive reason. Euclid, however, is wanting in many mathematical respects. He unfortunately separates two dimensions and three dimensions and he omits conics, but most important, he bases arithmetic on geometry. Euclid's methods are now altogether too cumbersome and too primitive. As a matter of fact, modern geometry per se begins with the concept of an unbounded line (not defined as the distance between two points). One thinks of variable points producing loci, and then of correlations between geometrical and algebraic forms, culminating in an arithmetical basis of analysis. R. Courant and H. Robbins begin their interesting book, *What Is Mathematics?*, with the statement, "Number is the basis of modern mathematics." No one would think of relying solely upon superseded results in science. Why, then, do we neglect the cumulative growth of mathematics?

Let us regard mathematics from a modern point of view.

These, I believe, are three fundamental axioms for teaching mathematical sciences: teach mathematics—with applications—from a modern point of view.

May I conclude with a few general remarks on the philosophy of education itself? I frankly confess that I am disturbed by two questions, and my misgivings undoubtedly arise largely from personal misgivings. In the first place, it seems to me that there is nowadays an overemphasis upon utilitarianism; and secondly, that the stated social goals of modern education may be somewhat meaningless without a compensating emphasis upon the individual.

In the sixteenth yearbook (1941) of the National Council of Teachers of Mathematics one reads "Since 1920 great stress has been placed upon the socialization of arithmetic." In examining a library copy of this book I ran across a 1946 newspaper clipping which noted, "Older children find . . . that their problems . . . are related to their own present and future lives." As examples of profitable problems, we find recommended the filling out of income tax reports and the determining of the cost of a trip from the east coast to the west coast.

In view of the general attitude of many education experts that children should be made more aware of the present world in which they live, it seems strange to find so many problems involving remote or even contingent use. Most of these efforts, moreover, are spent in acquiring merely materials and information, but very little if any on mathematics per se. Look at these recommended topics: taxation, banking, investment; reasons for price changes; basis of credit; computation of freight rates; secondhand buying; *et al.* All these may be of considerable importance for everyday living, and as such they are undoubtedly worth while. As far as I am concerned, however, they hardly constitute mathematics.

This modern utilitarian approach stresses subject teaching in striking contrast to the liberal medieval concept of teaching tools that may be valuable for many subjects. It is as if one were to learn a language merely to be able to order a meal—a worth-while objective, but hardly of great cultural value. Recently in the *Saturday Review* someone noted caustically that the early American colonists must have been greatly handicapped because they did not have available a book on "How to Be a Pioneer." Practical success may be achieved it seems, not with practical experience, but with theoretical guides.

Opposed to this present utilitarian socialization of arithmetic is the ageless ideal of liberal or general education. In the seventh chapter of Plato's *Republic*, happily translated by Jowett, "On Shadows and Realities in Education," one reads "A free man ought not to be a slave in the acquisition of any kind of knowledge, do not use compulsion, but let early education be a sort of an amusement." Plato was objecting here to the study of numbers by future merchants and retail traders, whose primary interest was their usefulness for buying and selling. Socrates says " 'I am amused,' I said, 'at your fear of the world which makes you guard against the appearance of insisting upon useless things.' " Aristotle says in his *Politics*, "If he desires it for the sake of its own development or with a view to excellence, it is liberal." There is a fifth freedom to which our children are entitled—the freedom to learn. We must not deny them intellectual enjoyment, their rightful heritage. We must not take the joy out of mathematics. Arithmetic is more than a social tool for adjustment to a social world. As such, indeed, it could be used for the enslavement of youth to their social environment. On the other hand, arithmetic can help youth adjust environment to a better and a freer life. Arithmetic can be—it should be—it must be fun!

Turning now to certain stated social

goals of education, I recently read this one given by a popular exponent of elementary science education, "The most important purpose might be phrased to help children gain the ideals, understandings, and skills essential to become good citizens." Who would deny such noble sentiments? Have not such hopes been in the breasts and minds of American teachers ever since the founding of our democracy? But who, pray tell, are the good citizens? And just what ideals, understandings, and skills do they need? The understanding and skill of mathematics? Apparently not. In the 1946 newspaper, noted above, we find that Miss X "told teachers that to learn all these things required a mental age of 15. She was speaking of long division." In this very Association you are aware of a recent movement to reduce the teaching of arithmetic in the elementary schools in order to transfer much of it to the secondary schools. The sixteenth yearbook of the National Council of Teachers of Mathematics echoes a related sentiment, "All too often teachers are disheartened when a superintendent desires goals such as co-operation, adjustment, happiness, while the parents expect the children to learn reading, writing, and arithmetic." Why not both goals? Are they incompatible? Indeed, may not the three Rs even be necessary both individually and socially to attain co-operation, adjustment, happiness? Science, indeed, looms up as one important element that we share in common with the Soviet Union, and as such it may be an international instrument for peacemaking.

In our goals of education, I believe we must be concerned not only with transitory appearances as seen by the men of a particular age, but also with the eternal structure of the universe as visualized by men of all ages. Regardless of our human impressions and mortal expressions, as long as the universe endures, mathematics will continue to contribute to our comprehension of it. Mathematics as *μάθημα* (learning) is here to stay!

Have you read?

CROSS, ROSAMOND, "The Basic Requirements of a Superior Education," *Journal of the American Association of University Women*, May 1956, pp. 218–221.

Here is another look into the future. You may disagree with it, but we should all be thinking about the problem. The author includes in the requirements the kind of education that will insure (a) steady growth in competence, (b) accurate vocabulary, (c) a critical faculty who challenges the student, (d) a parent who puts proper value on education, (e) a mastery of the basic understandings and learnings.

You ask the question, does this mean we are overemphasizing the identical opportunity for all? Read the article—but make up your own mind.

LESAGE, THE HONORABLE JEAN, "Alexander Graham Bell Museum: Tribute to Genius," *The National Geographic Magazine*, August 1956, pp. 227–256.

Don't let this title deter you because you feel you are familiar with the works of Dr. Bell. Here is information of special interest to the student of mathematics which is new to most of us.

The tetrahedron is a common regular solid, but where is its practical use? You will be surprised to read about the inventions of Dr. Bell which were based on this simple figure; for example, man-carrying kites, shelters, towers, and aircraft. Your students may be interested in the Bell Museum opened August 18th at the Bells' former summer home in Baddeck, Nova Scotia. It too is of tetrahedral motif both inside and out. Some of Bell's original notes are reproduced in the triangular glass panels.

This article contains many other items of interest. Be sure to read it and recommend it to your students.—PHILIP PEAK, *Indiana University, Bloomington, Indiana*

Toward discovery and creativity

NORTON LEVY, *Concord High School, Concord, Massachusetts.*

The true education of American youth and the real function of a teacher extend far beyond teaching the solutions of a quadratic equation. These involve the realm of the spirit and the nature of man.

MUCH ATTENTION has been given in the past few years to the shortage of teachers, especially in the sciences and mathematics. Publicity about and personal contact with the financial and occupational difficulties of teachers have diverted many promising teacher-trainees into seeking their future in other fields. Of four of my friends who were trained to teach mathematics, one is now in the aircraft industry, one is in electrical engineering, one in mechanical engineering, and one is now a statistician. As well-meaning personal associates ask me why I want to remain a teacher, adding that there is not much of a future in teaching, all I have to do is relate some recent happenings in my classroom to more than justify my position. What follows may reveal some of the main sources of satisfaction that I have found to exist in teaching mathematics.

The search for knowledge, in terms of a student's own emotional and intellectual leanings, can be influenced but not always controlled by his teachers. For instance, today Chris was wading through the theory behind the construction of a hexaflexagon, a paper folded into triangular sections in such a way that for each new twist a different hexagon will appear. Yesterday he found a set of directions and mechanically followed them. His satisfaction in the results and the interest of some of his friends encouraged him to pursue the topic further. Tomorrow he may be curious about other geometric forms.[1] Some day in the near future he may even make a significant contribution in this or some related section of mathematics. More likely, by next week he will be curious about some altogether different topic.

In the past few years Chris has been groping, discovering, and occasionally creating in more areas of mathematics than any preconceived course could possibly envision. For him, pursuit of mathematical knowledge has been worth-while primarily for its own sake. Chris may be endowed with special ability, but this doesn't explain his attitude. His natural ability, if such a thing exists, we can affect but little, it seems. What develops from his ability we can influence: (1) either positively, by providing situations with unpredictable possibilities for inquiry, and then being ready whenever signs of interest appear; or (2) negatively, by parrying his questions with such remarks as "wait till you get to college for this"—or even worse, by restraining him from showing extra interest through our failure to discourage the possible attitude of his peers towards (in their language) "his being a brain."

Once, when Chris was in grammar school, his arithmetic teacher informed his mother that he just didn't have too much ability along mathematical lines. Perhaps spurred on by this challenge,

[1] Actually, the following day Chris designed a Christmas greeting involving a hexaflexagon.

possibly struck by some mysterious mutational force, or more likely encouraged by later influences in his environment, Chris started to reveal his "not too much ability" by his sophomore year. When he happened to become aware of a nomographic method for associating days of the week with calendar dates, he applied a similar method to the Pythagorean Theorem. He prepared a device for determining the hypotenuse when given the legs of any right triangle. At another time he tackled a problem his geometry teacher had never had any of his pupils solve in seventeen years of teaching.[2] After several days of persistent effort Chris solved it.

In his junior year, because of other obligations, he was unable to attend regular meetings of the mathematics club. However, he managed to learn about each day's discussion after school. Once, when I introduced some ideas about prime numbers to the club, some student asked how we know the number of prime numbers is limitless. At that time I was unprepared for the question and was unable to guide the group toward an answer, except for mentioning a reference. When Chris appeared that afternoon and learned about the question, he quickly presented a proof he had seen somewhere in his readings.

Not long after I recommend a book in connection with a topic Chris finds intriguing, he may be expected to question some point in the book, which more likely than not is presented somewhat vaguely by the author. For example, Chris was recently disturbed by the apparent glibness of George Gamow's presentation of Cantor's proof that the integers cannot be placed into one-to-one correspondence with the decimals between zero and one.[3] Chris believed that by considering the digits of any decimal in reverse order the corresponding integer would appear. However, study into the nature of irrational numbers showed him this was not always possible.

Chris is now a senior. Not only does he have a wealth of experience in mathematics, much of it unconventional for a high school student, but he has developed a strong personal concern for the subject. He may never discover anything entirely original, but for Chris himself mathematics has become a real adventure with frequent revelations.

One way the beginning of a pupil's interest is indicated is in his noticing similarities or differences in the routine features of the subject. In a "slow" algebra class the topic under consideration was the multiplication of polynomials. Ellen noticed that the rows were indented to the right rather than to the left as in numerical multiplication. This led to the relation between the two types of multiplication and the arbitrariness of ordering the steps in the process. I wondered if the difference in the customary forms of indentation in arithmetic and algebra went back to the countries of their development. Wasn't algebra largely organized in countries whose languages read from left to right, while arithmetic came from countries where the reverse order of writing is used?

The expanded viewpoint resulting from his contact with a new idea can intensify a student's desire to investigate further. About the same time he was exposed to literal numbers in the eighth grade, Steve tried to uncover general relationships inductively. He wanted to simplify combinations of arithmetic operations. He listed pairs of numbers and their answers under various operations and then inferred what he could from the list.

In one case he considered the difference between the product of two numbers and their sum. The only time he noticed anything special was when the two numbers were equal. He found that

$$(a \times a) - (a + a) = a \times (a - 2).$$

[2] To construct a right triangle, given the radii of the inscribed and circumscribed circles.

[3] George Gamow, *One Two Three—Infinity* (New York: The American Library of World Literature, Inc., 1954), pp. 30–31.

Otherwise, when the two numbers were unequal, he could reach no generalization.

Steve also wanted to know what happened to two numbers when their sum was multiplied by itself various numbers of times. When he tried to find out what had to be added to the sum of the squares of two numbers to equal the square of their sum, he used different operations on the two numbers until he found that twice the product worked. However, although $(a+b)^2=a^2+b^2+abm$, when $m=2$, he could find no m such that $(a+b)^3=a^3+b^3+abm$.

When it came to adding and subtracting fractions Steve felt that he was more successful. Whenever such fractions as a/b and c/d were combined by addition or subtraction he found $(ad+bc)/bd$ equaled the result. When I commented that what he had done corresponded with what he had learned in earlier arithmetic, he said that at least now he understood why the process fitted the situation. Apparently the student's need for clarification is one of the main incentives for further study.

The connecting link between curiosity and discovery is strengthened by creativity. One way to stimulate the development of the student's creativity is by means of original problems. How often do we present problems having prearranged methods of solution ourselves and, lo and behold, one student offers an altogether different method and another presents a solution without even indicating his method.

An unusual work problem was passed on recently from *The Mathematics Student Journal*[4] to my classes as "the problem of the week." Chris defined several involved unknowns, formed equations, and eventually arrived at the solution. Steve, now a freshman, but unfamiliar with simultaneous equations, put together some half-imagined relationships which also led to the answer. With assistance he later managed to express his steps logically, but his original method involved mental activity more primitive and more abbreviated than formal verbal expression.

Another example was the well-known counterfeit coin problem,[5] which took me more time to solve when I first heard it than I'd care to admit. Joan, a freshman, solved it in one day using a method easier to follow than my own. The next day, John, a sophomore, presented a third method of solution in concise, essay form.

Joan recently read an article in THE MATHEMATICS TEACHER,[6] which deals with changing words to numbers that check under one of the operations of arithmetic. At about the same time, she learned about number systems to bases other than ten. Shortly thereafter she put the new ideas together and devised word-digit problems to the base five and to the base two.

I wonder if the true worth of the high school mathematics curriculum depends upon the exact contents of what is taught. Even though it may sometimes be necessary to delay or even to sacrifice some of the regular topics, the teacher who presents opportunities for boundless discoveries, who capitalizes on the unexpected, the one who stimulates inquiry as well as conventional learning, is more likely to foster the development of that type of confidence and independence which is closely allied with a love for learning.

Emerson went so far as to maintain that "A man should learn to detect and

[4] Vol. III, (October 1955), p. 4, problem 50: Two cows in four weeks can eat all the grass on two acres and all the grass that grows on these two acres in four weeks. Three cows in two weeks can eat all the grass on two acres and all the grass that grows on these two acres in two weeks. How many cows will it take to eat in six weeks all the grass on six acres and all the grass that grows on these six acres in six weeks?

[5] A counterfeit coin differs in weight from eleven good coins which have equal weights. Whether the counterfeit coin is lighter or heavier is not known. Explain how to locate the counterfeit coin, no matter which one it is, using a balance three times.

[6] William E. Beeman, "Originality in Arithmetic," THE MATHEMATICS TEACHER, XLVIII (November 1955), 495–496.

watch that gleam of light which flashes across his own mind from within, more than the lustre of the firmament of bards and sages."[7] Whether "from within" or from without, if a spark of original insight or creative expression appears in one of our pupils, we can help it to glow. If such expectations seem unrealistic for ninety-nine per cent of our pupils ninety-nine per cent of the time, we can be patient. Then when a chance for a student to express himself in an original way arises (and this chance can be helped along), we can treat it as an everyday occurrence, and afterwards it may be just that—an everyday occurrence.

[7] Ralph Waldo Emerson, *Essays* (Boston: James Munroe and Company, 1841). p. 37.

Have you read?

ROGERS, HARTLEY JR. "A General Education Course In Pure Mathematics," *The American Mathematical Monthly*, August–September 1956, pp. 460–465.

Wait! don't pass up this article because of the title. True, it is a description of a course given as a part of general education in Harvard University. Even so, this article contains much for the teacher of secondary mathematics and his students. It points up the contribution pure mathematics makes to the educated person. It gives us perspective of the mathematical structure. It presents ideas for discussion which could well be a part of the mathematics given in the secondary school. Such things are presented as the productivity of rigorous proof, false proofs and gaps in Euclidean geometry, the axiomatic approach, the place of assumptions, the concept of collection, and simplest ideas of rings or groups. As these are discussed, one sees many places where high school mathematics might well touch some of these topics and thereby make a greater general contribution. You will want to look over the reference list—it has some good suggestions for your library.

SATTERLY, JOHN. "The Circumcircle, the Incircle, the Nine-Points Circle and the Circle Through the Feet of the Bisectors of the Angles of the Triangle," *School Science and Mathematics*, October 1956, pp. 517–528.

That day when your class in plane geometry is completing the study of the orthocenter, circumcenter, and incenter of a triangle, and some of your not-so-good-students need further help, is a good time to turn your students who are gifted in mathematics to this article. Mr. Satterly not only gives again those relations which we all know, but through careful construction and proof points out others. He then suggests some relations which might exist, but which he has not yet isolated. Your gifted students will be happy to follow his steps and show that the incircle and nine-points circle are tangent at the Feuerbach point, that the circle through the feet of the angle bisectors passes through the Feuerbach point. His illustrations show what happens when the given triangle is changed.

This kind of a summary does not do justice to an article such as this which is packed with possibilities. I would say all those plane geometry students interested in higher mathematics should study this article.

THIRRING, HANS. "The Step From Knowledge to Wisdom," *American Scientist*, October 1956, pp. 445–456.

This is really a philosophical article and it is one that inspires, frustrates, and motivates to action in a single presentation. Would you agree that age is not essential to becoming wise, but that possession of the knowledge and good will for a better understanding and insight into ourselves and fellow men is essential? That the wise must also possess a good sense for proper order of values, duties, and responsibilities? Do you agree that the above abilities are attainable by the high school mind? How much knowledge is needed to attain this aim? Could we delete five per cent of what is now in the curriculum to provide time for gaining wisdom? Can we select that small amount of knowledge needed for teaching wisdom if it is taught? What about the internal contrasts of man? Would this gain in wisdom automatically eliminate the cultural lag? Would it decrease the suffering from avoidable causes which are now a blot on our civilization?

These and many other questions on the few simple laws of human nature are discussed. You will profit from reading this.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Symbolic logic and logical circuitry in the high school

STUDENTS OF ALGEBRA II, *University High School, Florida State University, Tallahassee, Florida.*

Foreign language teachers contend that children learn a new language more readily than adults. Could this be true of mathematics and its language?

FOR THE LAST HALF of this year, our second year algebra class left the mathematics of two hundred years ago and explored some of the new, more modern systems. The principal object of our attention was what is known as "symbolic logic."

The class was open to all students who had taken Algebra I, but only six students, (three seniors, two juniors, and one sophomore) chose the course. This allowed our instructor, Mr. J. R. Hooten, to give more individual attention than is usually the case. Because of this extra amount of help, the class made rapid progress through the material usually included in two semesters' work and completed its study in one semester.

Planning the content for the remainder of the year, we chose logic when offered several new subject matters for possible study. Mr. R. S. Fouch, Associate Professor of Mathematics Education at Florida State University, was invited to help teach, and was with us throughout the project.

We started our work by reviewing methods of proof. Mr. Fouch explained that when one is working with and discussing the meaning of a sentence, one can say the sentence repeatedly very easily. In recording the work it becomes very tedious to write the sentence over and over. This is where the symbols in symbolic logic come in. We substituted symbols for useful words in the sentences, such as "and," "or," and "not." We then symbolized phrases in the sentences by letters such as P, Q, and R. In this way, while discussing the sentence "We study and it is not true that we are late," we would refer to it and write it as "$P \wedge (\sim Q)$" "P" symbolizing the "we study," "$\wedge$" symbolizing "and," "$\sim$" symbolizing "it is not true that," and "Q" symbolizing "we are late."

We examined many types of sentences and soon realized that some sentences had the same meaning although stated and symbolized differently. We learned that whether two sentences were the same or different depended upon their "truth value." If the truth values (whether the sentence was true or false) of two sentences were the same under all conditions, the two sentences were alike or equivalent. Our consideration of the truth value of sentences composed of more than one phrase led us to the study of "truth tables."

If a statement, such as "$P \wedge Q$," has a definite number of separate components, each of which is either wholly true or

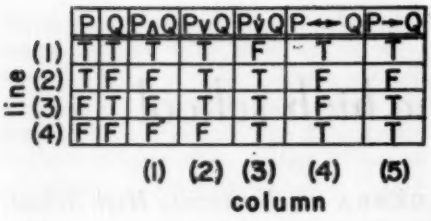


Figure 1

wholly false, the truth value of the statement can be shown by a "truth table." Figure 1, Column 1, shows a truth table, the one for the form statement "$P \wedge Q$" (statement P and statement Q). Line 1 is read, "If statement 'P' is true and statement 'Q' is true, the statement '$P \wedge Q$' is true." Line 2 is read "If statement 'P' is true and statement 'Q' is false, the statement '$P \wedge Q$' is false." Since the statement "$P \wedge Q$" contains two variables ("P" and "Q") and each has two variations (truth or falsity), the statement can exist under 2·2 or four conditions, which are represented by the four lines of the truth table. Likewise, a statement with three variables would have eight lines in its truth table. We did not consider statements with more than two possible variations in truth value.

Most of our time in the first part of the project was spent studying the way two sentences or phrases are combined and how this affects the truth value of the combination. The operations (ways to combine sentences) we studied were:

Column 1. "and" ($\wedge$) "I will go to town *and* you will go with me."
Column 2. "or . . . or maybe both" ($\vee$) "I will buy him a present *or* I will give him money *or maybe both*."
Column 3. "either . . . or" ($\downarrow$) "*Either* you will work *or* you will be fired."
Column 4. "if and only if" ($\leftrightarrow$) "You will be paid *if and only if* you do the job."
Column 5. "if . . . then" ($\rightarrow$) "*If* you ask me, *then* I will help you."

We devised truth tables for the first four operations with a minimum of effort, but the "if . . . then" stumped us for a while. Going to the example, we could see that if "you asked me" and "I helped you," the statement would have been true, and if "you asked me" and "I didn't help you," the statement would have been false. If "you didn't ask me," but "I helped you anyway," would the statement have been false? By regarding the statements as parts of a proof, we discovered the statement would have been true.

We entered the second phase when Mr. Hooten slyly asked us if we thought a statement's being true or false could be related to an electrical circuit being closed or open. With this question, our project took a new tack, for we soon realized that the truth value of a sentence containing one of our five operations depended on the truth value of its clauses in much the same way as the flow of current depends on whether switches are open or closed. We decided that it would be both beneficial and fun to try to build electrical circuits that would express truth values much as truth tables do. We decided to use a light bulb to indicate the truth value of the complete sentence (on for true, off for false) and switches to indicate the truth values of the substatements or variables.

Our search for circuits proceeded in this way. Looking at the "$P \wedge Q$" truth table, we could see that to represent it we wanted a circuit with two switches ("P" and "Q") and a light ("$P \wedge Q$") arranged so that when both switches were closed (true), the light would be on (true); if either or both switches were open (false), the light would be off (false). Our "$P \wedge Q$" circuit is shown in Figure 2. In like manner we arrived at the "or . . . or maybe both" circuit (Figure 3). In both of these circuits we used single pole-single throw switches using closed for the true and open for false.

For the "if and only if" (Figure 4) and "either . . . or" (Figure 5) circuits it was necessary to use single pole-double throw switches, with one closed position for true and another closed position for false.

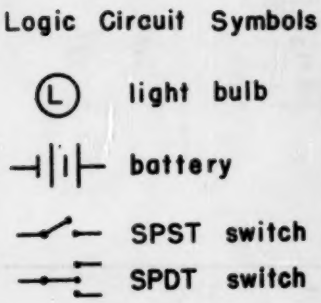
We noticed that the "if . . . then" statement was similar to the "or . . . or

maybe both" since if "*P*" was true the statement was true in both cases. The difference was that in "or . . . or maybe both" if "*Q*" was *false* the statement was *true* (Figure 1, Column 5). So for "*P*→*Q*" we made "closed" on the "*Q*" switch be "false" and "open" be "true." Thus we arrived at the circuit in Figure 6.

After building the circuits theoretically, we tried them out with electrical parts and they worked satisfactorily. We had built truth tables out of wires, lights, and batteries.

Looking for new worlds to conquer, we realized that there was a limit to the number of truth tables for combinations of two variables. For a statement with two variables, the truth table had four lines, each of which could be either true or false. Four variables, each with two variations, yielded sixteen combinations, and we had only explored five. The other eleven were quickly and easily found and labeled.

To complete this phase of the project we worked on combining the sixteen different circuits into one "thinking machine." (It was to be for two-variable statements only. We did not attempt to construct three-variable circuits.) To simplify construction, we divided the sixteen into eight pairs, each pair containing two "opposites"—tables in which the "trues" and "falses" are interchanged. (Columns 3 and 4 would make a pair.) In this way we were able to construct only eight circuits and a "negation circuit" which would change a "true" to a "false." We would set up a circuit for Column 3, then for Column 4 would open the same circuit, but reverse the answers with the "negation circuit."



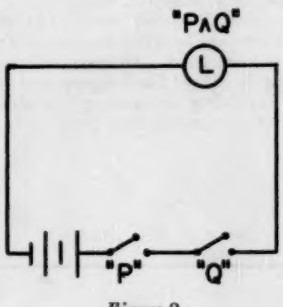


Figure 2

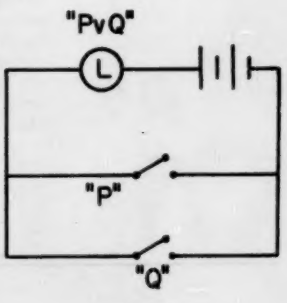


Figure 3

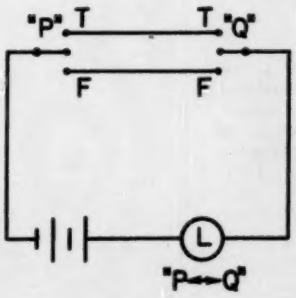


Figure 4

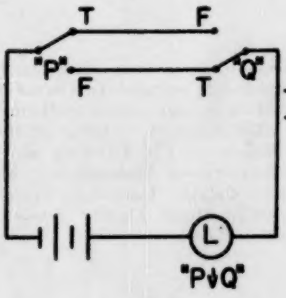


Figure 5

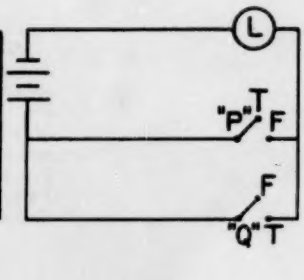


Figure 6

After completing the circuit, we boxed it attractively and dubbed it the "Thimac."

We received much favorable publicity for the project, including an article in a state-wide magazine and the appearance of Mr. Hooten on television to demonstrate the "machine."

Along with and after our symbolic logic study we explored permutations and combinations, mathematical odds in gambling, topology, and, at the same time this is being written, the theory of sets.

Chief among the benefits of our study was the valuable experience in working in a new mathematics, with strange symbols and new operations. We have learned much about the logic behind our everyday language, and we hope that through our contact with logic we can develop more critical and analytical minds.

ARCHIE FREEMAN
JEFF HOWARD
VIRGINIA GUNN
EDNA FREEMAN
LARRY STEYERMAN
WILLIAM BOYCE

The pupil who solves a difficult problem in brokerage may have the pleasant consciousness of having overcome a difficulty, but he cannot feel that he is mentally improved by the effort he has made. To attain this end he must feel at every step that he has a new command of principles to be applied to future problems. This end can best be gained by comparatively easy problems, involving interesting combinations of ideas.—*Committee of Ten*, 1892.

Methods of teaching statistics need to be developed and then made part of the professional preparation of high school mathematics teachers.—*This statement is based on the study by Max Beberman.* The Teaching of Statistics in Secondary School Mathematics, *Ed.D., 1952, Teachers College, Columbia University, New York City; Major Faculty Adviser, Professor Howard F. Fehr.*

Homework and achievement in plane geometry

VYNCE A. HINES, *University of Florida, Gainesville, Florida.*

Some more light is cast on the difficult problems pertaining to homework in the high school.

AFTER SURVEYING the related literature and research on homework, Ruth Strang comments, "Although many opinions exist on the effect of homework on scholastic success, there has been little research on the subject."[1] Such research has often been of short duration and has lacked adequate controls. This study is an effort to answer the question of the value of out-of-class study for achievement in plane geometry.

MATCHING OF GROUPS

Two classes were formed using the matched-pairs technique and included all pupils who had registered for plane geometry. The experimental group did no home study; the control group did do home study. Pairing was done on the basis of chronological age, intelligence quotients, and point averages for two semesters of beginning algebra. The groups were later compared on scores on the Co-operative Achievement Tests in Algebra and in Plane Geometry which were given during the first days of the experiment. The groups were found to be quite similar on these tests, but these data were not used in forming the original pairs. Nineteen pairs were formed, and the two participating teachers took turns drawing from each pair for members of their respective classes.[2]

Peters and VanVoorhis' Formula 95 was used to calculate standard errors of differences between means in terms of the differences between paired measures.[3] The differences found were small ones and could have occurred by chance from forty-five to ninety-seven per cent of the time. Sixteen of the nineteen pairs finished the year and were used for the experiment, and for calculating the matching results just reported. Standard deviations were comparable for the two groups. Both groups had a higher mean IQ—120.1 and 119.2—than would be found ordinarily in a high school class. However, IQs ranged from 91 to 147 in the experimental group and from 81 to 144 in the control group, in the original nineteen pairs, and from 99 to 144 in both groups among the sixteen pairs who finished. The algebra test scores indicate a high level and almost equal retention over the summer by the two groups. The initial plane geometry test scores indicate very little knowledge of plane geometry and for many of the pupils the scores were accountable by chance.

[1] Ruth Strang, *Guided Study and Homework.* ("What Research Says to the Teacher," No. 8) (Washington, D. C.: National Education Association, Department of Classroom Teachers and American Educational Research Association, 1955).

[2] The writer taught the no-home-study group; Miss Mary Iball taught the home-study group. The study was made several years ago at the University High School, University of Illinois, Urbana, Illinois.

[3] Charles C. Peters and Walter S. VanVoorhis, *Statistical Procedures and Their Mathematical Bases* (New York: McGraw-Hill, 1940), p. 165.

Teaching and testing

The two teachers agreed to follow the same textbook[4] closely. The book was divided into eight units. An objective test booklet accompanied the text. At the end of each unit, both groups took this test on the unit. The following day, beginning with the end of the second unit, pupils took a cumulative review test, also from the booklet. The teachers planned their work so that each group spent the same number of days on each unit covered.

The experiment ran from September until June, approximately 180 school days. Teaching procedures were kept as uniform as possible by the two teachers. Both groups met at the same hour, 8:10 A.M., for a class which lasted a full sixty minutes. Both groups devoted part of most periods to supervised study, so that often the control group had its outside study started before it left class. Both groups were taught by student teachers for identical periods during the second semester—teachers who received identical grades from the same three judges. The principal difference between the two groups was that in the control group homework was assigned two or three times a week. While records were not kept, it was thought that the average pupil would take from forty minutes to one hour for assignments.

The nature and purpose of the experiment were explained to both classes at the beginning of the year and the resulting pupil co-operation was thought to be excellent by the two teachers.

Students from the control group worked slightly faster than those in the experimental group on the unit tests and cumulative review tests. More pupils from the control group were able to finish these tests during the sixty-minute period than were able to finish from the experimental group. Whether this was because of extra, outside-of-class practice is not known.

At the end of the first semester both groups took an objective test prepared by the high school mathematics staff. At the end of the year both groups were tested on the Co-operative Plane Geometry Test, Form N—the same test administered at the beginning of the year.

For the purpose of the experiment, the only evaluation attempted was that which measured achievement in plane geometry as measured by the scores on unit tests, cumulative review tests, a semester examination, and a standardized geometry test.

It might be pointed out that if there is an advantage in out-of-class study, then the design of the experiment meant that any "cheating," i.e., study outside of class by members of the experimental group, or failure to study outside of class by members of the control group, should serve to reduce any differences found between the two groups.

Absences from the two classes were about the same. Any make-up work was done in class by members of the experimental group during supervised study periods. Neither group received appreciable out-of-class help from the teachers. So far as is known, no additional tutoring was done outside of class for members of either group.

Experimental data

Means, standard deviations, differences of means, critical ratios, and the probabilities of the critical ratios were determined for the two groups on the eight unit tests, seven cumulative review tests, the semester examination, and the Co-operative Achievement Test in Plane Geometry.[5] Peters and VanVoorhis' Formula 95 was used to calculate the standard errors of differences between means in terms of the differences between paired scores. This formula makes unnecessary the tedious calculation of correlations between matching variables, here, multiple *R*s.

Every one of the seventeen comparisons

[4] Virgil S. Mallory, *New Plane Geometry* (Rev. ed.; Chicago: Benjamin Sanborn, 1943).

[5] A tabular presentation of these statistics can be obtained from the author.

TABLE 1—Comparison of Experimental (E) and Control (C) Groups on Co-operative Achievement Test in Plane Geometry

	Initial Test		Final Test		Gains		Difference in Gains
	E	C	E	C	E	C	
$\bar{X}$	35.5	35.8	55.6	61.1	20.1	25.3	5.2
s	8.9	7.0	10.5	7.4			
D	.3		5.5				
t	.10		1.50				1.55
p	.92		.16				.15

favored the control group. Nine of the differences are statistically significant at or beyond the five per cent level, three others at or beyond the ten per cent level. Variability of performance tended to increase in both groups on both the unit tests and the cumulative review tests. During the first semester—Units I through IV—the experimental class tended to be more variable; during the second semester, more variability was observed in the control group. Although no strong trend is apparent in the size of *ts*, the size of the differences between the groups increased. During the first semester the mean differences were 4.0 and 2.4 on the unit and review tests. During the second semester the mean differences were 5.0 and 6.2, increases in spread of 1.0 and 3.8 points.

Differences on the Co-operative Achievement Test give approximately the same t value whether they are calculated on differences in final scores or differences in gains. These scaled-score differences are equivalent to about a 25-percentile spread. Translated into letter grades, the typical experimental-group pupil made a C, while his control matchee received a B.

Conclusions

1. Out-of-class study, usually written work, increases achievement in plane geometry.

2. The differences in achievement tend to be cumulative.

3. Differences tended to be slightly greater on cumulative review tests than on unit tests covering recent material.

4. If a traditional grading system were used—A, B, C, D, E—and if students were graded only on the tests reported here, home study would increase the grade of the average student by one letter.

Limitations

It is possible that the differences found could be accounted for by differences in skill or zeal on the part of the two teachers involved. Hence, it would be desirable to repeat the experiment with other classes and with the home-study situation reversed.

While the groups were well matched, they were small. Further, they were somewhat superior in intelligence to most plane geometry classes. These factors make it a little hazardous to generalize the findings to larger and less able classes. Further, the small number of cases makes it futile to study differentials by intelligence levels.

The experiment does point out, however, a design by which future studies can obtain more adequate research from which to generalize. It is suggested that pairs of teachers in many schools use this design in classes in different subject areas, those in which homework is primarily written and those in which assignments emphasize reading or other activities. It would be well to accumulate results for groups of varying ability and also for classes of different sizes. A few dozen such studies would give definitive answers to some questions in the homework versus no-homework argument.

The functional approach to elementary and secondary mathematics

WILLIAM A. GAGER, *University of Florida, Gainesville, Florida.*

There are those who feel that adolescents will like mathematics if it is meaningful to them. Many feel that "making mathematics functional" is the best way to make mathematics meaningful.

MODERN-DAY EDUCATORS believe that our public schools should strive to build up those *meanings* which will best develop the total personality of our boys and girls. Also, almost everybody believes that our public schools, even though they cannot do all of this job of building meanings, can do more of it and can do it better than any other one organization. On the basis of these two ideas it seems unfortunate indeed that so much of our school time for mathematics is spent in presenting facts, practices, and drills, while so little time is spent in developing and clarifying the meanings of mathematical ideas and principles.

We are seriously searching these days for an answer to this question: How can the public school curriculum in mathematics be organized so that it will give proper emphasis to *the meanings* which it has to offer to our young people? One quick way to answer this question is to say that the mathematics curriculum should be planned to meet pupil needs. The big objection to this quick answer is that to plan broadly enough to meet pupil needs is a problem complicated enough for an Einstein.

Who knows what a pupil needs? In this connection I might mention the little girl who complained to her father that her mother did not know much about bringing up children. The daughter based her criticism on the fact that her mother made her go to bed "when she was wide awake" and made her get up "when she was awfully sleepy."

Surely there is a lot we do not know and never will know about pupil needs. But this we do know: Pupil needs are certain to be tied up with experiences that reach into the past, and pupil needs cannot possibly be met without giving careful thought to the demands of the future, which is the adult viewpoint. Even though it greatly confuses the issue, the facts in the case force us to admit that the everyday needs of our pupils are strongly tied in with the past and with the future. Thus, it would not be wise for us to ignore either the traditional or the adult viewpoint as we plan our mathematics courses to meet the needs of our pupils at the various grade levels.

We often hear it said that a pupil need is *a felt desire for something* that comes directly from the pupil. However, we must not overlook the fact that a pupil need is often a desire which the pupil becomes conscious of and concerned about because of certain developments and understandings generated by the curriculum material. Also, a pupil may become conscious of a need that has gradually been developed by the clever planning and action of a good teacher. In all probability the development of many essential needs would never come from the pupil if he were left to build on his own limited ex-

perience. Would you be one to contend that genuine pupil needs cannot be generated by carefully selected materials? Or would you be one to claim that it is not the main business of a teacher to stir up a realization of certain needs in the minds of pupils?

Much more could, and probably should, be said about pupil needs, but let us move on to a consideration of subject matter. Down through the years it has been common practice to describe our different school curricula in terms of subjects or subject matter. Both on the elementary and secondary levels there have been some experimentations with *broad areas* of study, but it has been felt by most educators that the experiences of youth can be adequately utilized in the building of *meanings necessary for a good life* within the framework of certain subject fields.

As an example, take arithmetic which happens to be one of the subject fields looked upon with favor by educators. There are those, of course, who feel that "arithmetic is arithmetic," the same yesterday, today, and forevermore. They feel that "it is what it is" and can never be changed. Such a conception of arithmetic is certainly biased. Of course, it is true that some arithmetic concepts, principles, and procedures change very little if at all, but arithmetic as a subject never has been and never will be an immutable body of knowledge.

The experiences of youth and our interpretations of the needs of youth are causing arithmetic to be continuously altered. There is a vast difference both in content and organization between *Pike's Arithmetic* published in 1788 or *Ray's New Intellectual Arithmetic* published in 1885 and the arithmetics published in 1955. Even though they differ among themselves, these arithmetics published in the last few years are good arithmetic books. Recognition of the fact that a pupil develops *meanings for living* only through past and present experiences which have meanings for him has caused authors to present in our modern-day arithmetics, in the best psychological patterns, those materials that will be most likely to develop meaningful situations. Thus, it becomes evident that the scope of the subject, arithmetic, as it exists in our present curriculum is not a narrow offering, but rather a "broad area" of study. In fact, educators are rapidly coming to see that a subject-matter field, such as arithmetic, can be made as broad as any other approach which they know how to make.

We realize, of course, that whether or not a subject is developed on a broad level depends upon whether or not authors writing in the field decide to write to reach the pupils or to expound the logic, laws, and technical aspects of the subject. If the authors do the latter, then only those pupils who can comprehend a high level of abstraction will be able to understand and absorb the materials. Little learning can take place for most pupils from texts that are too abstract.

Because of certain difficulties encountered in trying to make functional those curricula that are built around subject-matter courses, some educators believe that it would be better to build public school curricula around *broad areas* such as health, physical education, practical arts, social studies, language arts, music, and science. In all such developments that I have seen, the assumption is that mathematics will be utilized in these broad areas as the need for it arises.

There is no question as to the importance of mathematics to the empirical sciences, the social sciences, and to a lesser extent all the broad areas of study mentioned. As is so often pointed out to us, mathematics is a science in its own right. It is a mode of thinking that grows step by step from one fact to another much like one brick is placed upon another brick in building a tower or a smokestack. Mathematics brings order out of chaos. It is a tool in one sense, but it is more than just a tool to be picked up when needed and then laid aside.

It is impossible for a pupil to pull out of thin air with any consistency that portion of mathematics which he needs at a certain spot in some particular *broad area of study* and expect it to function for him, unless he knows his mathematics and has had some experience in applying it. *The very nature of mathematics and the numerous purposes it serves has made all efforts to fuse it into a core curriculum totally ineffective. On the basis of our past experience the following conclusion seems inescapable: If in the future the elementary and secondary curricula for our public schools are built on the broad area plan, then mathematics should be one of these broad areas, and this broad area of mathematics should be taught only by those who have been adequately trained to teach in such an area.*

Past experience indicates that it would be far more effective to bring social, scientific, and other fields of thought into the mathematics courses rather than to attempt to teach mathematics at such times as there appear needs for it in other areas of study. This idea is sound provided it is worked out by good teaching from teachers who are well trained in mathematics. It is not sound when teachers fail to tie the mathematics to pupil experiences in a manner that will build durable life meanings or when teachers fail to teach the mathematics materials so that they can be fully comprehended by pupils. Not to be able to understand a situation is irritating and sometimes quite ridiculous.

Efforts to remedy some of the shortcomings and mistakes of the past in preparing materials and in teaching elementary and secondary mathematics have brought to the forefront a developmental type of presentation known as the *functional approach*. In this developmental, or functional, approach these facts are of prime importance:

1. Pupils learn only those facts which have meanings for them, that satisfy them in some way, or that they feel will be of importance to them in the future.
2. Mathematics must provide experiences that are personally satisfying and socially significant, as well as mathematically sound, if it is to contribute successfully to pupil learning.
3. The theory and practice involved in providing the basic ideas and procedures of mathematics must be presented in a manner that will stimulate maximum insight and understanding.
4. Materials must be made available in mathematics textbooks that will give pupils experience in using mathematics both as it applies to everyday affairs and as it applies to taking the next steps up the mathematic ladder. This does not mean that all mathematics materials should be of a strictly practical nature. It simply means that the mathematical principles and procedures should be presented in a manner that will serve some useful purpose in the thinking of the pupil.

Insofar as we can successfully work out the preceding steps, we will be moving toward functional teaching, a type of teaching that almost entirely eliminates the rote-teaching of facts, practices, and drills.

Rote-teaching of facts, drills, skills, and knowledge, as a means of establishing understandings and meanings in mathematics, should be a thing of the past. Facts taught in isolation with the hope that they will take their proper places in problems when they are needed is mostly a waste of time and energy. We know from experience that isolated facts will not transfer to problems to help pupils when they need them. We also know that drills given before a pupil fully understands the meaning of a mathematical idea or principle, can do the pupil more harm than good. This does not mean that practices and drills are not important. It means that they are not important until after the pupil has obtained a full understanding of the ideas on which he is to be drilled.

Certainly "rote" and "telling" types of teaching, which have been so prevalent in our classrooms in the past, have had much to do with making pupils hate elementary and secondary mathematics. By using suitable functional materials and by training ourselves to be expert in the functional type of teaching, it is hoped that we will be able to eliminate much of this hatred and other unfavorable attitudes towards mathematics and save much time and talent.

In mathematics the functional approach in teaching usually begins with an overview of what is to be done, followed by an inductive- or scientific-method type of development. This inductive type of reasoning, which as you know was the first scientific method used in the search for truth, is still the best approach we know anything about to enable us to tie in with the personal experiences of our pupils. It leads, step by step, from known experiences to new experiences in such a way that there builds up within one's thinking an urge to formulate a generalization. Surely, pupils who have been trained and encouraged to formulate principles by the inductive type of thinking are the ones who will most likely be able to apply the more formal mathematically sound deductive-type proofs to their inductive generalizations.

From a geometry class comes this very interesting example of functional teaching. To this class the teacher, Miss Hart, spoke these words: "Surveyors, contractors, real estate men, power-line men, boat-race men, and others often need to know the size of angles in a triangle. Many times, too, they need to know the sum of the interior angles of a triangle. Between now and tomorrow I would like to have you try to find some way of finding the sum of the interior angles of a triangle. If these two geometric facts, which you already know about, can be of any help to you, use them:

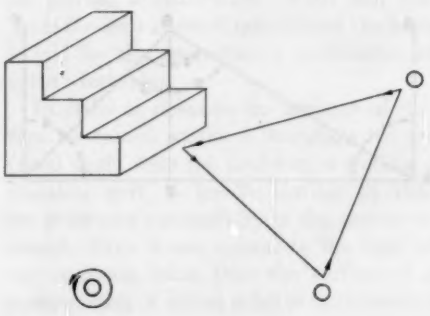
1. A perpendicular drawn to a line forms right angles.
2. When two parallel lines are cut by a transversal the alternate-interior angles are equal."

On Tom's back porch steps sat three members of Miss Hart's geometry class. Said Tom, "If I walk over to that tree on my right, face left, and walk over to that next tree and again face left and walk back here, I will almost be facing in the opposite direction (Fig. 1). On the other hand, if I walk completely around a tree I will turn through 360° and will be facing the same way I started. If walking a complete circle makes me face all the way around and walking a triangular path makes me face half way around, it looks to me that the sum of the interior angles of a triangle is $\frac{1}{2}$ of 360° or 180°."

Dick suggested that they draw several triangles of different shapes and measure the angles of each triangle with a protractor and add them. Using ten different triangles, the boys found the sum of the angles for each triangle to be 180°.

At this point, Harry suggested that he would get his father's vernier protractor and measure the angles of the triangles with it. Using the vernier protractor the boys found that the sum of the angles for some of the triangles was a little more and for some a little less than 180°. They were quite pleased, however, when they added the sums of the ten triangles and averaged them to find that they got 180° for each triangle.

Figure 1



"It certainly looks as if the sum of the interior angles of any triangle is 180°, but can we be absolutely sure about this?" asked Harry. "You certainly can't," replied Dick. "You have to be awfully careful about reaching conclusions. For example, I was always certain that there was no such bird as a white robin until one built a nest in that tree last year."

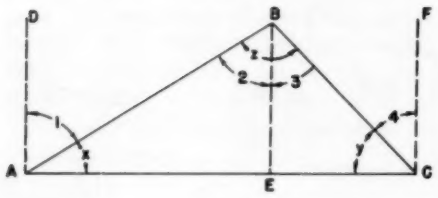
In school the next day the boys explained why they felt that the sum of the interior angles of a triangle is 180°, but not one of them could prove that this statement was true. So Miss Hart suggested that each member of the class draw a triangle and from the left and clockwise around it place the letters A, B, and C at the angle points. Then she asked them to construct the lines AD, BE, and CF perpendicular to the side AC (Fig. 2).

"Can you give us one true relationship from your drawing, Tom?" asked Miss Hart. Tom replied, "Since $\angle DAC$ is a right angle by construction $\angle 1+\angle x=90°$, and for the same reason $\angle ACF$ is a right angle and $\angle 4+\angle y=90°$. Thus, $\angle x+\angle y+\angle 1+\angle 4=180°$."

Dick came in at this point with the facts that $\angle 1=\angle 2$, alternate-interior angles, and for the same reason $\angle 3=\angle 4$. This makes $\angle 1+\angle 4=\angle 2+\angle 3=\angle z$.

Harry finished the proof by saying, "Since it has been proved that $\angle x+\angle y+\angle 1+\angle 4=180°$ and $\angle 1+\angle 4=\angle z$, then it must follow that $\angle x+\angle y+\angle z=180°$. Eureka! We have proved that the sum of the three angles of a triangle is 180°."

Figure 2



This example of functional teaching shows how the boys arrived at a generalization by induction. Then by deductive reasoning, one of the most powerful methods available to the mathematician, they proved that the sum of the interior angles of a triangle is 180°.

As an example of the functional type of teaching in another field, we call your attention to the football coach. Even though the coach realizes that the boys do not know enough football to run the formations, block, and so on, he usually calls for a scrimmage the first week of practice. Terrible though this first scrimmage may be, the coach knows that this is the best way to get the boys ready to play the games. Through scrimmage, the boys quickly and jarringly come to realize that there are some further things about football that they need to learn. In a very real sense the scrimmage creates a learning situation. It puts the boys in the right mood to absorb the coach's "know how" and to perfect certain necessary understandings and skills. Thus, there follow cycles of scrimmages which give the overall picture, instructions which develop understandings of things to be done, and drills to make the boys skillful in carrying out their assignments.

The big objective in the functional type of teaching in mathematics, or any other field of study, is to make contact with the personal experiences of the pupils in such a way that they will see some value in what is to be done and get the feeling that they will learn something worthwhile by doing it. Only when pupils have been brought to see the sense of a thing and some good reason for learning something about it will the interrelationship between pupils and teachers be functioning in such a way as to develop "life meanings."

Formulas can be fun

J. MAURICE KINGSTON, *University of Washington, Seattle, Washington.*

Learning is the act of structuring experiences.
Teaching is the act of aiding others to structure experiences.
This paper is rich in suggestions for helping junior high-school children to structure experiences involving area.

ONE CURRENT MEANS of introducing high-school students to algebra is through the use of the common mensuration formulas, a method which I believe has considerable merit. However, there is a very real danger that many students may not understand the significance of these formulas nor be aware of the ease with which these mathematical truths can be demonstrated.

In what follows, the emphasis will be on the *sequential* approach, the building of one stage upon another, which I feel is so consistent with learning procedures in mathematics at all levels. I hope this treatment will serve also to give students a feeling for the dimensionality of each concept so that not only will the formulas have meaning but there will be less tendency to use an incorrect exponent, for example, in powers of the radius of the circle or sphere.

The devices which I shall discuss are described in many arithmetic texts, e.g., in Robert Lee Morton's *Teaching Children Arithmetic*,[1] and in Burdette R. Buckingham's *Elementary Arithmetic*.[2] However it is their relative dependence on each other which I am most anxious to point out here.

Throughout the elementary grades children have been encouraged to measure and work with lengths. It seems to me that the most important feature associated with this concept is the necessity for having a *unit* of length with which to work. This is essential if our operations are to convey any meaning whatever to another person. Within reason any unit is employable. The determination of the particular unit to be used, however, is usually dictated by the distance to be measured. Once this choice has been agreed upon we can proceed to make informative statements in terms of this unit. I am going to assume, for the purpose of this discussion, that the concepts of perimeter and area of a rectangle, as well as the volume of a rectangular box, have become familiar to an average eighth-grader and thus our beginning point, the circumference of a circle, makes definite contact with his past mathematical experience.

I would have a class compute the ratio of circumference to diameter in a number of circles, using string and the familiar tin can. When these results have been assembled it will not come as a shock to them that this famous ratio has been given a special letter, π (pi), in order to insure its eternal identification. With the formula $C = \pi d = 2\pi r$ well established the basis is laid for the approximate verification of future formulas.

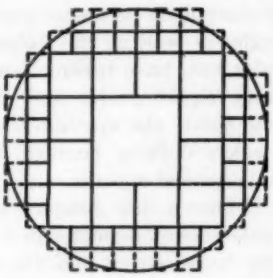
In order to describe the amount of surface contained within a boundary we are faced again with the problem of finding a suitable unit, a hurdle similar to that we overcame successfully in the matter of length. Does it not appeal to the logic of our common sense that the surface of a square, each of whose sides is the accepted

[1] Robert Lee Morton, *Teaching Children Arithmetic* (Chicago; Silver Burdett Company, 1953), Chap X.

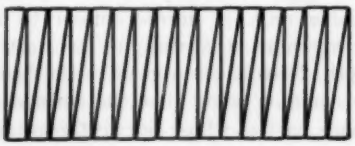
[2] Burdette R. Buckingham, *Elementary Arithmetic: Its Meaning and Practice*, (Boston: Ginn and Company, 1947), Chap. XV and XVI.

unit of length, will do excellently as the unit of area, or one square unit? Only by having a clear understanding of the unit of area, can the product of the length and width of a rectangle be truly meaningful as the measure of its area.

When we attempt to measure the two-dimensional space within a curved boundary, like a circle, the situation is not as simple. The very existence of a unique area is, indeed, a matter for question. In order to obtain an intuitive answer to this problem we may equip ourselves with a number of paper squares in the form of square units, quarter square units, sixteenth square units, and so forth, and use them to cover the space within the circle. As we work closer and closer to the circle's edge from both inside and outside we can see that it is possible to bring the maximum and minimum approximations for the area as close together as we wish and it becomes reasonably plausible to accept the existence of a unique "area."



To establish the formula for the area of any circle, I like to think of the circumference as divided into a large, even number of equal arcs. When the sectors thus obtained are laid alternately adjacent to each other, and their number is allowed to increase indefinitely, the resulting figure approximates more and more closely a rectangle with sides r and πr, giving πr^2 for the required area.



It is also possible to consider the area of the circle from another point of view, that of interpreting the previous sectors as approximate triangles of altitude r and with the length of each arc as base. The product of radius and semi-circumference again produces the above formula for the area of the circle.

Knowing the circumference of a circle, it would be no more difficult to determine the number of square units in the curved surface of a right circular cylinder one unit high than to investigate the area of the four faces of a closed, rectangular fence one unit high. Hence the area of the curved surface of a cylinder of radius r and height h units must be given by the product of circumference and height.

It is now possible to investigate the surface area of a sphere and one method is to use a rubber or wooden ball cut in half. If a tack is inserted at the center of the diametral plane, a light weight cord of proper length can be laid continuously in a spiral around the tack to cover the space within the circle. If a similar procedure is carried out on the curved surface of the hemisphere, it is found that the cord required to accomplish the latter is approximately twice as long as the first cord. Hence the fact that the surface of the sphere is *four* times as large as the area of the diametral circle should no longer present any disturbing mystery. Again let us emphasize that we are not pretending to prove anything but are merely intending to make the results reasonable, attractive, and meaningful and to provide the students with a means of recalling them quickly to memory. At this point some bright student may be encouraged to make the remarkable discovery that, if a right circular cylinder is circumscribed about a sphere, the curved surface of the cylinder will have the same area as the sphere.

Before leaving the topic of surface area, let us note that a right circular cone can be cut along an element and the surface then laid out flat on a plane. If r represents the radius of the base and s the slant height of the cone, then, when flat, the surface covers a sector (with an arc length $2\pi r$) of a circle of radius s. Using the same procedure of alternate adjacent sectors, as in the case of the circle, we obtain a rectangle with sides s and πr, whose area is πrs. Alternatively we have here an elegant opportunity to illustrate the use of the principle of ratio to obtain the desired result.

In the hope that the subject of area has been made more palatable by the above sequential treatment, let us turn our attention to the matter of volume. In exactly the same manner that the unit of area was deduced from the unit of length, we can now choose, as our unit of volume, the space within a cube of height equal to one unit of length, and base one square unit of area. The use of such a cubic unit makes it comparatively easy to introduce the formula for the volume of a rectangular box. But if the volume of such a box is evidently the product of its base and its height, then it follows, logically, that the volume of a right circular cylinder must be obtained by multiplying the area of its base by its height.

It is not difficult to construct a cardboard cone to match the base and height of a cylindrical tin can. Using sand as the medium, students can repeat for themselves the satisfying experiment of filling the cone *three* times in order to fill the cylinder once.

The analogous treatment for the sphere may be accomplished by cutting a hollow rubber ball in two equal parts. The next step is to construct a cone with height and base radius both equal to the radius of the hemisphere. The final discovery to be made is that *two* cones of sand will fill the hemisphere. At this point it is very instructive to obtain a cylinder with radius and height also equal to the radius of the hemisphere. The volumes of the cone, hemisphere and cylinder will then demonstrate the ratio 1:2:3.

It may also be advantageous to note that the same volume formula which holds for cones must hold, by a reverse limit argument, for pyramids. Accordingly, if we were to consider the surface of the sphere divided into very small squares, say, and the edges joined to the center of the sphere, we would have decomposed the sphere into a sum of near-pyramids all having the radius of the sphere as altitude. Therefore the product of one-third of the sum of all the bases (one-third of the surface of the sphere), by its radius, would again provide the expression for the volume of the sphere.

In conclusion, we should observe a common feature associated with all our area formulas, whether area lw of a rectangle, s^2 of a square, πr^2 of a circle, $4\pi r^2$ of a sphere or πrs of a cone; they all involve *two* variables, thus demonstrating their *two-dimensionality*. On the other hand the volume lwh of a rectangular parallelopiped, s^3 of a cube, $\pi r^2 h$ of a cylinder, $\frac{1}{3}\pi r^2 h$ of a cone and $\frac{4}{3}\pi r^3$ of a sphere, all reiterate the fact that such volumes are consistently *three-dimensional*.

It is my hope that, by making the above verifications for themselves, students may be led to see very early some of the elegance and orderly arrangement of facts which is such an important and compelling attribute of mathematics.

Trigonometric functions of real numbers

JOHN A. TIERNEY, *United States Naval Academy, Annapolis, Maryland.*

The trigonometric functions of real numbers are of greater importance in mathematics than the trigonometric functions of angles. Why not make the change in both high school and college courses?

IN A RECENT ARTICLE[1] Richard V. Andree has discussed the desirability of introducing the concept of a trigonometric function of a number. In this paper I would like to suggest a simple method of accomplishing this purpose and to indicate some of the advantages to be derived from it in elementary trigonometry and calculus.

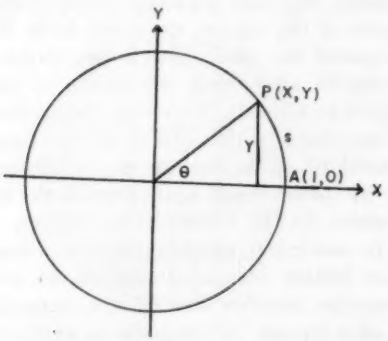


Figure 1

In Figure 1, θ is an arbitrary angle in standard position and AP its intercepted arc on a unit circle with center at the origin. Positive and negative angles are measured in the usual fashion. We now set up a one-to-one correspondence between the set of all angles and the set of real numbers. With each angle θ we associate the directed distance s through which the point A moves as the angle θ is generated.[2] For example 0° corresponds to 0, 30° corresponds to $30/360(2\pi) = \pi/6$, $-90°$ corresponds to $-\pi/2$, 540° corresponds to 3π, etc. At this point the student should work a large number of exercises. Given an angle such as 15° he should be able to find from a rough sketch that the number associated with it is about 0.25 and given a number such as 0.4 that the angle associated with it is about 23°. The important point is that the association of angle and number should become natural for the student. This would require time and effort but would result in clearer future understanding.

We notice that the word radian has never been mentioned and that no special significance has been attached to the angle associated with the number 1. From similar figures it is now shown that if θ intercepts an arc S in a circle of radius R, then $S = Rs$ where s is the real number associated with θ. Since s is thought of as an abstract number the dimensionality of this equation should not cause difficulty.

We now define the sine of θ or the sine of s as the ordinate of point P in Figure 1. Similarly the other trigonometric functions are defined as functions of angles or as functions of real numbers. Drill would be given in approximating trigonometric functions of various angles and various numbers from a graph. Since we admit

[1] Richard V. Andree, "Modern Trigonometry," THE MATHEMATICS TEACHER, XLVIII (February 1955), 82.

[2] Length of a circular arc is discussed in books on plane geometry.

that the analytical aspects of trigonometry are far more important than the computational, we might well argue that the trigonometric functions should be defined primarily as functions of real numbers. The dual definitions suggested here, however, would permit the usual development of computational trigonometry.

We now consider the graph of the function $y=\sin x$ where x and y are real numbers. From Figure 1 it is evident that we wish to show how Y varies with s.[3] It thus seems natural to use the same scale on the x and y-axes and to indicate multiples of $\pi/2$ on the x-axis. These are clearly the values of s for which we can find Y by inspection.

The graph of the inverse function $y=\sin^{-1} x$ shows how s varies with Y and the function is seen to be multiple-valued. The equation could be read "y is a number whose sine is x" and principal values would be defined in the usual manner. The older notation $y=\arcsin x$ is very suggestive here since s is actually an arc whose sine is Y.

The graphs of functions like $y=x+\cos x$ now become more meaningful because the student need never think of an angle. Calculus books note that in expressions like $\sin x$, x is measured in radians unless otherwise stated, but unfortunately this seems so unnatural to the student that he is often confused.

In order to differentiate $y=\sin x$ we would present one of the well known arguments that

$$\lim_{t\to 0} \frac{\sin t}{t}=1.$$

[3] It should be emphasized here that it is merely customary to denote the dependent variable by y and the independent variable by x.

Most students will anticipate from Figure 1 that

$$\lim_{s\to 0} \frac{Y}{s}=1.$$

We can also point out that s need not approach zero through positive values alone. We thus obtain the result that the derivative of the sine of a number is its cosine. Then, when we develop the Maclaurin expansion of $\sin x$, it will seem just as natural as the expansion of the exponential function.

We now consider the following problem from the integral calculus:

Find the area bounded by the x-axis, the y-axis, the line $x=1$, and the curve $y=1/(1+x^2)$.

Applying the Fundamental Theorem of the Calculus we seek a function of x which is single-valued on the unit interval and whose derivative with respect to x is $1/(1+x^2)$. We obtain

$$A=\int_0^1 \frac{dx}{1+x^2}=\tan^{-1} x\Big|_0^1$$

and frequently the student is mystified because he does not see how an angle is related to the area under the given curve. (It is difficult enough for him to see that the area is computed by using only two values of $\tan^{-1} x$.) Using the approach suggested he would read the above as "the principal number whose tangent is 1 minus the principal number whose tangent is 0."

In addition to eliminating some of the difficulties encountered by students of trigonometry and calculus, the above method leads more naturally to the definitions of trigonometric functions of complex numbers.

An easily constructed chart for finding the roots of quadratic equations[1]

GEORGE BARNES, *Humboldt State College, Arcata, California.*

High-school pupils may be interested in the construction and use of a chart to solve certain quadratic equations.

THE REAL ROOTS of any quadratic equation with rational coefficients may be read from a simple chart which can be constructed on ordinary rectangular co-ordinate paper by means of a straight edge only.

The chart is constructed by plotting on the co-ordinate paper the pairs of values which p and q, of the quadratic equation $x^2+px+q=0$, assume if one of the roots takes on a constant value, say b. (It will be remembered that $-p$ is the sum of the roots, and q is their product.) The result of such a plot is found to give a different straight line for each different value of b.

For simplicity the discussion and development of the method will be limited to positive roots; the extension to include negative roots will be obvious. Consider first the case where both roots are integral, and one of the roots has the value $x=1$. The other root, and also the product q of the roots, will be one of the numbers in the sequence $1, 2, 3, \cdots, n, \cdots$, and the sum $-p$ of the roots will be the corresponding number in the sequence $2, 3, 4, \cdots, (n+1), \cdots$. A straight line is obtained by plotting the q's and corresponding p's, which in this case are the points $(1, 2), (2, 3), (3, 4), \cdots, (n, n+1), \cdots$, on rectangular co-ordinate paper with $-p$ along the vertical axis and q along the horizontal axis. This is one of the lines on the graph shown in Figure 1. The slope of this line is $+1$ and its y-intercept is $+1$, therefore its equation must be

$$-p=q+1.$$

Other straight lines shown in Figure 1 are obtained by giving one of the roots other integral values. If $x=2$, $-p$ takes on the values, $3, 4, 5, \cdots, (n+2), \cdots$, and q has the corresponding values $2, 4, 6, \cdots, 2n, \cdots$. The straight line obtained by plotting these q's and corresponding p's may also be found in Figure 1. Its slope is seen to be $+\frac{1}{2}$ and its y-intercept $+2$. Its equation is then

$$-p=\tfrac{1}{2}q+2.$$

In the general case, where one of the roots is given the value b, $-p$ then assumes the values $(b+1), (b+2), (b+3), \cdots, (b+n), \cdots$ and q takes on the corresponding values $b, 2b, 3b, \cdots, nb, \cdots$. Upon plotting the q's and corresponding p's for this case, it is seen that the slope of the straight line which results is $+(1/b)$ and its y-intercept is $+b$. Then its equation is

$$-p=\frac{1}{b}\,q+b, \tag{1}$$

where b may be either positive or negative. Moreover, b may also be fractional. By drawing in a large number of such lines and including positive and negative values of both p and q, it will be seen that there is a parabolic region of the qp-plane

[1] Editor's note: For a similar chart and a more extensive discussion of the graphic solutions of equations see T. R. Running, "Graphical Solutions of the Quadratic, Cubic, and Biquadratic Equations," *The American Mathematical Monthly*, XXVIII (November-December 1921), 415–23.

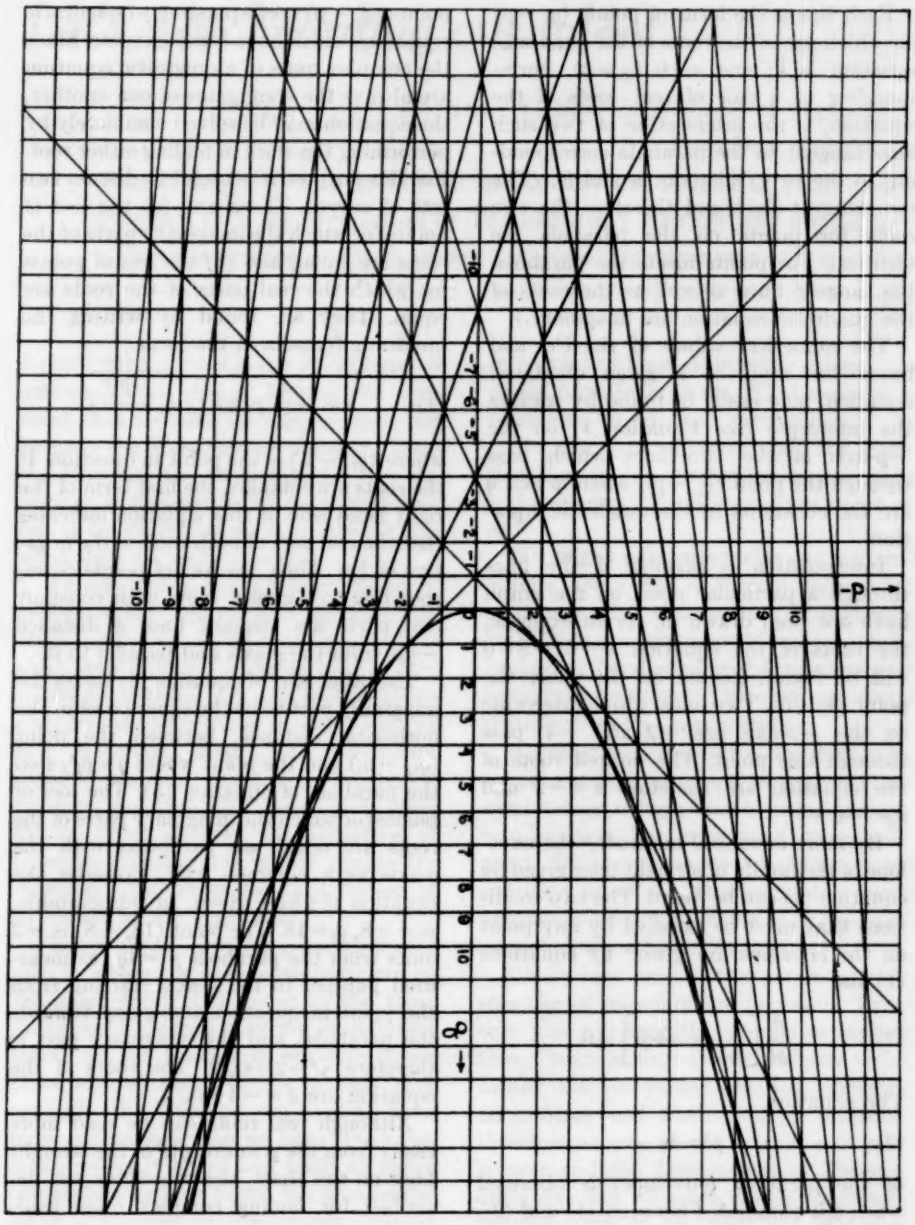


Figure 1

through which no lines pass. The parabola is the envelope of the straight lines. Of course all the lines are tangent to the parabolic envelope. The parabolic region contains values of p and q corresponding to imaginary roots of the quadratic equation, and these roots will be discussed presently.

Each line is the locus of points $(q, -p)$ for which one of the roots of the quadratic equation is b; and each $(q, -p)$, corresponding to a pair of real roots of the equation, is the intersection of two such lines tangent to the parabola corresponding to the two real roots b_1 and b_2. (The two tangent lines and therefore the two roots for points on the parabola are identical. For points inside the parabola, the tangent lines as well as the roots of the quadratic equation are imaginary.)

The numerical values of the b's, and hence the roots of a given quadratic equation, may easily be found by reading the intercepts (see Equation 1) on the $-p$-axis of the two lines which pass through the point $(q, -p)$, where p and q are the constants in the quadratic equation.

Interpolation is possible if the lines through a particular point on the graph have not been drawn in. As an example, the roots of the equation $x^2+6x+8=0$ will be found. Locate on the chart the point $(8, -6)$. Two lines, whose intercepts on the $-p$-axis are -2 and -4, pass through this point. The desired roots of the equation are therefore $x=-2$ and $x=-4$.

By more advanced methods,[2] the envelope of the family of straight lines given by equation (1) can be found. The two conditions that must be satisfied by any point on the envelope are given[3] by equations (1) and

$$\frac{\partial}{\partial b}\left[\frac{1}{b}\,q+b+p\right]=0. \tag{2}$$

The equation

$$p^2=4q \tag{3}$$

of the parabola (envelope) is obtained when b is eliminated between (1) and (2).

As mentioned before, the part of the plane on the right of the vertical axis which is enclosed by the parabola contains points $(q, -p)$ corresponding to quadratic equations which have complex roots. Since the complex roots of a quadratic equation are always the conjugates of one another, the equation may be solved completely by performing the work of finding either root. For this purpose it is useful to discuss two sets of curves. These are: (a) the loci of points for which the imaginary parts of the roots are equal, and (b) the loci of points for which the real parts of the roots are equal. They are found by writing the quadratic formula in the form

$$x=-\frac{1}{2}\,p_1\pm i\sqrt{q_1-\frac{{p_1}^2}{4}}, \tag{4}$$

where $(q_1, -p_1)$ is the point in question. If the roots are complex, the first term of the right hand side of this equation indicates that the real part of both roots is the negative of $\frac{1}{2}p_1$. Thus, the loci of points corresponding to complex roots with constant real parts are straight lines a distance $-\frac{1}{2}p_1$ from the q-axis and parallel to it.

The other part of equation (4) shows the imaginary part to be the square root of the horizontal distance, between the point $(q_1, -p_1)$ and the point where $q={p_1}^2/4$ on the parabola of equation (3). The loci of points for which the imaginary parts of the roots are equal are parabolas with the q-axis as a common axis. Consider the equation $x^2+8x+18=0$. In this example, $p_1=-8$, $q_1=18$. The point $(18, -8)$ is -2 units from the parabola $p^2=4q$ (as measured parallel to the q-axis starting from the point in question and going towards the parabola) and the imaginary part is therefore $\sqrt{-2}=i\sqrt{2}$. The roots of the equation are $x=-4\pm i\sqrt{2}$.

Although real roots can be read more easily from the p intercepts of the straight lines on the chart, the procedure just described for finding complex roots lends itself to finding the real roots as well. It must be remembered that the numerical sign of the distance from $(q_1, -p_1)$ to the point on the parabola where $q={p_1}^2/4$ as measured in the positive q direction determines whether a root is real or complex.

[2] Agnew, R. P., *Differential Equations* (New York: McGraw-Hill, 1942), 61–63.
[3] *Ibid.*

Let's stop hoarding our talent

LEE A. DU BRIDGE, *California Institute of Technology, Pasadena, California,*

This article appeared in The Saturday Review, *August 4, 1956 and is reprinted with the permission of* The Saturday Review *and the author.*[1]

This penetrating analysis of our "shortage" of engineers and scientists should provide educators with food for thought.

I FEEL THAT a good many Americans are getting a little sick of hearing the argument that *we* must do this, or that, just because the Russians are doing it. What the Russians are doing may be very important in deciding what we should do—but to conclude that we must always *copy* what they do may be fatal.

We don't send thousands of people to concentration camps just because the Russians do. Nor do we deprive farmers of their land to fulfill a social theory. And we do not deprive men and women of the comforts of daily living in order that all raw materials, labor, and productive capacity may be devoted to building a military machine. In these and other things which destroy the initiative or freedom or happiness of the people the Russians, you might say, are "ahead of us"—and we are glad of it!

"Being ahead" implies some kind of a race in which the two contestants are on the same track and going in the same direction. If instead they are on different tracks or are headed in opposite directions, who is to say which is "ahead?" For example, the Russians are said to have more submarines than we have. Does this mean we must hurry and build as many submarines as they have? *Not necessarily!* Their submarines are presumably directed at our absolutely vital sea traffic. Their sea traffic may be relatively small and unimportant to them. Hence our submarines would have little to do. What we need is something to kill their submarines.

We hear over and over again the assertion that in the production of scientists and engineers "the Russians are ahead of us."

We might ask—what else can an ambitious young man or woman prepare for in Russia other than science or engineering? There is no great need there for stockbrokers or bond salesmen! For lawyers or bankers or preachers! There are not many opportunities for new business.

We must be conscious of the fact that we cannot build the kind of a society we want unless enough smart youngsters go into business, law, economics, or government. Whatever we do to obtain more scientists must not be done at too great expense to other vital fields or to the freedom of choice of the individual.

Let us turn now to the nature of this shortage of scientists and engineers. I am sure you will hear the opinion expressed that, "since a large fraction of our technically trained people are now engaged on military and atomic-energy projects, therefore as soon as the present cold war is over and the Department of Defense quits spending so much money there will be thousands of engineers out of jobs—and they will again be a glut on the market as they were in 1932."

It is to me perfectly obvious that in ten or twenty years, barring a world catastrophe, we will need twice as large a *frac-*

[1] The full text of Dr. Du Bridge's article will appear in the 1956 edition of the *Age of Science*. This excerpt is published with the permission of the editor of *Age of Science*.

tion of our working force engaged in science, engineering, and medicine as we have today. The task will be gigantic.

If we could improve the efficiency of utilization of the 600,000 men and women already trained and at work in engineering—by only five per cent this year—we would thereby add to the effective engineering force as much as all the new graduates of the class of 1956! Can such a thing be done? I think it could—with effort.

In the first place, since shrewd foresight is a great American virtue it is natural for every industrial manager to save up a few dollars for a rainy day—also a few tons of steel or copper or aluminum or oil or whatever he might need. Why not at the same time stash away a few engineers too? They'll come in handy if that next big contract does materialize. And, besides, they can be put on the payroll of the other Government contracts in the meantime and it doesn't cost the company a dime to hang on to them!

How much of this goes on, if any? No one can prove a thing. But many an employed engineer will tell you that this is happening. Besides, it is not illegal—and is indeed just a good example of that old-fashioned virtue of thrift! Only a program of education and propaganda can help, possibly with contract penalties for unreasonable hoarding practices. Companies who have to pay for their own engineers will use them effectively.

Many engineers complain that companies often fail to recognize the difference between engineers and draftsmen. As a result, hundreds of engineers' desks are jammed side by side in warehouse-type buildings amid all the clutter and clatter of typewriters, computing machines and jangling telephones. We must never forget that the kind of engineering we are short of is not routine drafting but the kind of thinking that is elicited fully only under the best physical conditions.

Industry is not the sole culprit. The Government too must take a great deal of blame for inefficiency and incompetent leadership. Furthermore, interservice rivalries and consequent duplication in the military-weapons program, obsolete and cumbersome methods of making choices between rival enterprises or devices, the lack of any machinery for stopping any project that is under way—all these things require ten engineers to do the work of eight or fewer.

I think I have said enough to illustrate the point. But this is only a short-range approach to our problem. It is at the level of the high school—and possibly even more in the sixth to eighth grade—that the really important increases in future supply of trained technical talent is to be found. And here again mere numbers are not the answer.

In the field of manual labor if one man can't lift a stone possibly two or three men can—or a machine can be found to do it. In the scientific or engineering field it is not assured that if one man can't solve a problem then two or three or more can solve it. And certainly no machine will of itself do it. Creative ideas occur in the minds of single individuals.

If the creative minds in high school can be encouraged to go to college—if we can give them more stimulating textbooks, if we can persuade their conselors that scientists are not just technicians who make terrible weapons, if we can bring parents out onto the frontiers of science to see the opportunities which exist for their children, if the whole community can learn to know the scientist not as an absent-minded professor intent on blowing up the world but as the man who has lifted civilization from Dark Age feudalism and slavery to twentieth-century liberty and enlightenment—and if the top 25 per cent, say, of the engineering graduates could be encouraged to go on to an advanced degree, with possibly 10 per cent of them pursuing a doctoral degree, the quality of the nation's research and development effort could be substantially improved.

The addition formulas

FREDERICK H. YOUNG, *Portland State College, Portland, Oregon.*

Teachers of trigonometry will be interested in the variety of ways they can derive the addition formulas.

IT HAS LONG BEEN recognized that the addition formulas are the heart of trigonometry. Indeed, Professor Rademacher and others have shown that the entire body of trigonometry can be derived from the assumption that there exist functions S and C such that

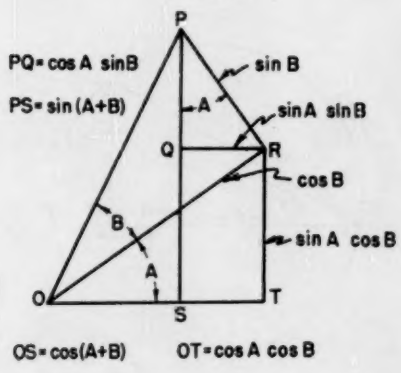
(1) $S(x-y)=S(x)C(y)-C(x)S(y)$

(2) $C(x-y)=C(x)C(y)+S(x)S(y)$

(3) $\lim_{x\to 0+} \frac{S(x)}{x}=1.$

It is not our purpose to go through this well-known development. Rather, we wish to demonstrate a number of different derivations of the addition formulas, showing the intimate connection of these formulas to a varied body of trigonometric and geometric material. Most of these derivations have appeared in texts or in journals. Several have appeared over the years in the *American Mathematical Monthly*.

Figure 1



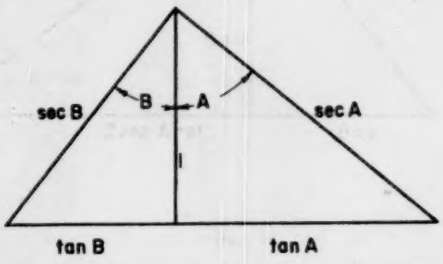


Figure 2

Our basic assumption will be that the trigonometric functions are defined as coordinates on the unit circle.

1. One of the most familiar derivations is based on Figure 1. The advantage of this method is that once it is pointed out that OP is unity and that angle QPR is A, the students can fill in the values of the remaining lengths and arrive at expansions for sin $(A+B)$ and cos $(A+B)$ for themselves. The objection to the method is that generalization to all angles is difficult.

2. A derivation based on the law of cosines is obtained from Figure 2. We have

$$\cos(A+B)$$

$$=\frac{\sec^2 A+\sec^2 B-(\tan A+\tan B)^2}{2\sec A\sec B}$$

$$=\frac{2-2\tan A\tan B}{2\sec A\sec B}$$

$$=\cos A\cos B-\sin A\sin B.$$

Again, generalization to all angles is a little involved.

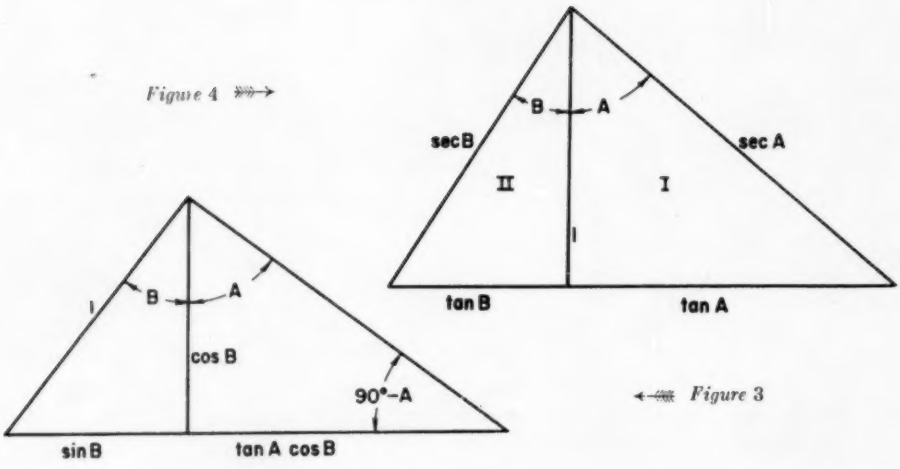


Figure 4

Figure 3

3. The law of sines can be used in much the same way, provided that sin $(90°-A)=\cos A$ is derived independently from the original definition of the functions in terms of the unit circle (Fig. 3).

$$\frac{\sin (90°-A)}{1}=\frac{\cos A}{1}=\frac{\sin (A+B)}{\tan A \cos B+\sin B}$$

$\sin (A+B)=\sin A \cos B+\cos A \sin B$.

4. Another method, only superficially different from the previous one, is based on area. Area of $\mathrm{I}+\mathrm{II}=(1/2)\sec A \sec B \sin (A+B)=(1/2)(\tan A+\tan B)$ or, $\sin (A+B)=\sin A \cos B+\cos A \sin B$ (Fig. 4).

5. An interesting "lo-and-behold" proof can be based on Ptolemy's Theorem that the product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of opposite sides. In a circle of diameter 1, construct angles A and B as in Figure 5. The construction lines show that the other diagonal is $\sin (A+B)$. Hence, the expansion for $\sin (A+B)$ is immediate.

6. A similar proof[1] uses the same circle of unit diameter and the same lemma that a chord in such a circle is the sine of the arc subtended (Fig. 6).

[1] See Householder, in *American Mathematical Monthly*, XLIX, 326.

Figure 5

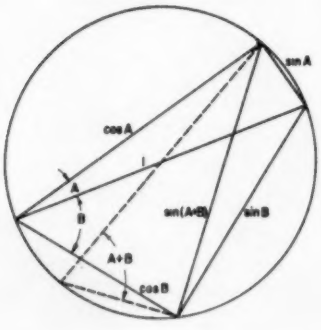
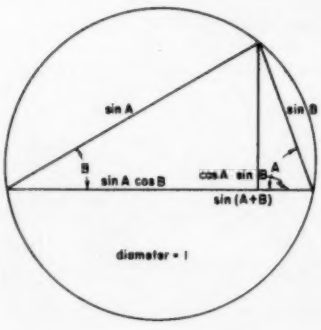


Figure 6



7. A simple derivation that was essentially given by Cauchy and that has the advantage, not shared by the preceding examples, of being general is based on the law of cosines and the formula for the distance between two points (Fig. 7).

$$d^2=(\cos A-\cos B)^2+(\sin A-\sin B)^2$$
$$=1+1-2\cos(A-B)$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

8. A similar proof, avoiding the law of cosines, can be based on the Pythagorean Theorem alone (Fig. 8).

$$d_1^2=d_2^2, \text{ or}$$
$$(1-\cos(A+B))^2+\sin^2(A+B)$$
$$=(\cos B-\cos A)^2+(\sin B-\sin A)^2$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

9. A variation of the preceding method is based on Figure 9.

Figure 7

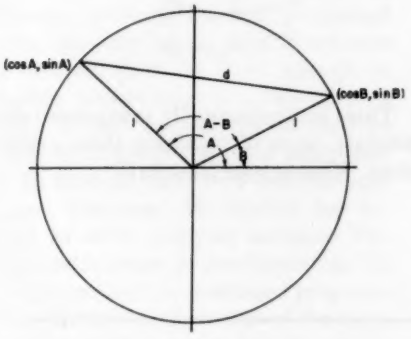
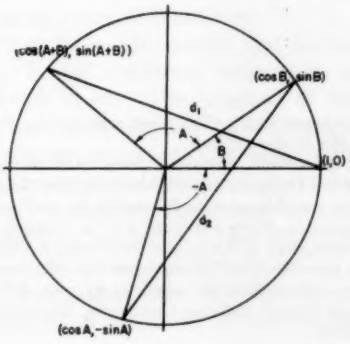


Figure 8



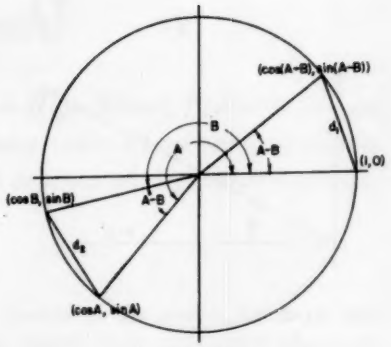


Figure 9

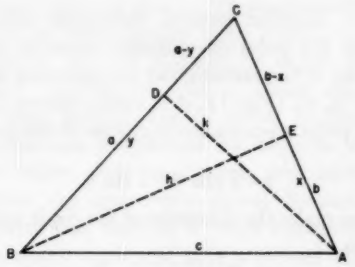


Figure 10

Equating d_1 and d_2, we get the expansion of $\cos(A-B)$.

10. If A and B are acute angles, the rather messy derivation below can be used. Here, AD is the altitude from A (of length k), BE is the altitude from B (of length h), $AE=x$, and $BD=y$ (Fig. 10). Then

$$h^2=a^2-(b-x)^2,\ k^2=b^2-(a-y)^2$$
$$c^2=h^2+x^2=a^2-b^2+2bx, \text{ and}$$
$$c^2=k^2+y^2=b^2-a^2+2ay, \text{ whence}$$
$$c^2=ay+bx=y\frac{h}{\sin C}+x\frac{k}{\sin C}, \text{ or}$$
$$\sin C=\frac{h}{c}\frac{y}{c}+\frac{k}{c}\frac{x}{c}$$
$$=\sin A\cos B+\sin B\cos A.$$

But

$$\sin C=\sin(180^\circ-(A+B))=\sin(A+B).$$

Again, this latter step follows from the original definitions of the functions.

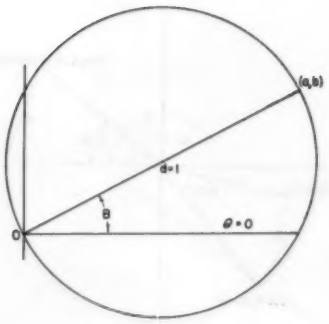


Figure 11

11. Another general derivation follows from the polar co-ordinate equation of a circle. If the rectangular co-ordinates of P are (a, b), (Fig. 11) it is easily shown that the polar co-ordinate equation of the circle is

$$r = a \cos \theta + b \sin \theta.$$

If we make the diameter of the circle unity, then

$$r = \cos B \cos \theta + \sin B \sin \theta.$$

However, another form of the equation of the circle is clearly

$$r = \cos (\theta - B).$$

12. A final general derivation is based on the formula for the distance from a point to a line. The equation of line (1) is $x \sin A + y \cos A = 0$ (Fig. 12). The distance from P to (1) is then

$$d = \sin (A + B) =$$

$$\pm (\sin A \cos B + \cos A \sin B).$$

If $A = 0°$, we have $\sin B = \pm \sin B$. Thus the $+$ sign must be chosen. Similarly, if (2) is normal to (1), it follows that

$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

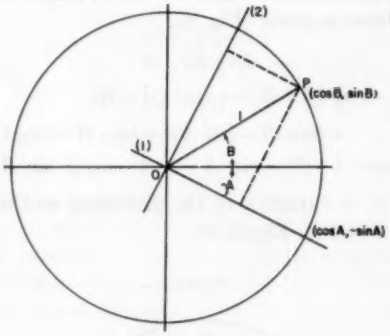


Figure 12

There are undoubtedly many more elementary ways of obtaining these expansions. What is your favorite?

Have you read?

WEAVER, WARREN, "Lewis Carroll; Mathematician," *Scientific American*, April 1956, pp. 116–128.

Alice in Wonderland is a household word to most of us, but few know the author, Rev. Charles Lutwidge Dodgson. His pen was very productive, having written over nine hundred items. Fifty-eight of these dealt with mathematics and logic. He wrote texts on arithmetic, algebra, trigonometry, plane geometry, and analytic geometry. The most serious was *Euclid and His Modern Rivals*.

This article, however, deals with the questions of Dodgson's proficiency in the field. It points out his accomplishments as well as his deficiencies. Your students will be interested in the discussion of his pillow problems, his barber shop paradox, and his notes on the infinite unit. They will be led to wonder, as was Alice.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

This isn't bonehead math!

ORVIN P. BARNEY, *Jefferson High School, Portland, Oregon.*

Refresher courses need carry no stigma.
Much depends on the teacher's attitude.

MANY SENIORS at Jefferson High School in Portland, Oregon, are taking a refresher course in mathematics, not because they must have it to graduate, not because they were programed for it against their will, but because they want it. Because they feel the need and desirability of the instruction, students are requesting that they be programed for MR, mathematics refresher.

Four years ago, in 1952–53, when a half-year course was initiated, students who were two years, or more, retarded in arithmetic fundamentals were programed for the semester class. End-of-semester tests showed such excellent growth in arithmetic computational ability that a full-year course was offered in 1953–54 and has been given each year since that time. In 1954–55 there were four full-year sections. This year the number has increased to seven full-year sections. The unexpected increase in enrollment can be attributed not only to continued programing in refresher mathematics of students who are retarded, but also to the increased popularity of the course. There is no longer a stigma attached to the course.

In 1952, when the course had its beginning, it was commonly referred to as "Bonehead Math." The teachers of the course, the counselors, and the administrators of the school all began an active educational campaign to remove the taint previously placed on the course by students and teachers alike. Today this stigma has been, in most part, replaced by a healthy realization that only those students capable of benefiting by instruction are allowed in the course. Students who have passed their arithmetic placement test, but who really do need the course, are encouraged to take it. Those students who do not need further basic arithmetic instruction are not allowed to enroll in refresher mathematics. And those students who cannot benefit from further instruction are programed for subjects or classes where they can more profitably gain from the instruction given. Because the course has become a more desirable course, many boys and girls who feel the need of "brushing-up" in arithmetic before leaving school are electing refresher mathematics voluntarily and are receiving worthwhile instruction.

A statistical evaluation of 65 students in three sections of the 1954–55 MR class at Jefferson High can well demonstrate the effectiveness and usefulness of refresher mathematics. The Hundred-Problem Arithmetic Test, Forms V and W,[1] were given to the students. One form was given in the fall before instruction began, the other form in the spring near the completion of the school year.

Test Score Mean	Before Instruction	After Instruction
(problems correct)	36.7	65.9
Standard Deviation	16.1	17.7

The mean (average) number of problems answered correctly at the end of the year was 29.2 greater than the mean at the beginning of the year. Upon subjecting this information to statistical analysis, it

[1] R. Schorling, J. R. Clark, and M. A. Potter, *Hundred-Problem Arithmetic Test* (New York: World Book Co., 1942).

was discovered that this difference was significant to the point that an increase of this magnitude could not possibly have happened by chance. Therefore, on the average, the students did significantly improve in arithmetic computational proficiency during the year.

The proof of the course, however, need not depend upon statistical analyses. Many students have expressed, with warm enthusiasm, their regard for the help they gained from the course.

At the completion of the final standardized test, one girl said she knew she had learned a lot about arithmetic during the year, regardless of how her test turned out.

Another girl, with a part-time job that required mathematics, said she had been helped a great deal by her increased understanding of arithmetic.

One graduate, who now works for a grocery chain store, says he thinks his refresher mathematics was the most important class he had in high school. He also expressed concern over the fact that the high school students he knew could not be persuaded to take their basic arithmetic fundamentals seriously.

Many students, boys and girls alike, have expressed their belief that they learned more, or at least received more benefit, from a refresher course in mathematics than they did from all the other mathematics they had. Even if this statement is somewhat discounted as excess exuberance of youth, it still speaks well for MR (mathematics refresher)

Mathematics for Decision Makers

Business executives recently have been flooded with descriptions of the wonders that can be performed with the aid of such devices or techniques as electronic computers, mathematical programing (linear programing and some aspects of operations research, for example), and servomechanisms or information theory. Some of these are the result of refinements in devices and techniques that have been used or known for a relatively long time. Others are relatively new. All of them require fairly extensive knowledge of certain branches of mathematics, on the part of *someone*, in order that they can be thoroughly understood and operated to advantage.

Does this mean that mastery of calculus, matrix algebra, probability theory, and topology is becoming a management prerequisite? *Definitely no.* Does it mean that the top executives of tomorrow must be better skilled in fundamental operations like calculating percentages and working out equations? *Not necessarily.* Does it mean that more understanding of the mathematical approach to problems will be useful to the man who makes important business decisions? *Unqualifiedly yes.*

The recent developments in this field do not change the basic character of the decision making that the businessman must do. Rather, they permit more rigorous formulation of the problems he faces, facilitate the figuring of possible situations for him to consider, and add explicitness and quantification to the terms in which he frames his decisions. At the same time, it cannot be denied that a top executive would benefit from knowledge of certain kinds of mathematics. This has always been true. The only difference is that now, because of the technical advances that have been made, the task of understanding and communicating is more difficult and the potential of usefulness is richer.—*By R. K. Gaumnitz and O. H. Brownlee, Taken from the* Harvard Business Review for *May–June* 1956.

Squaring a circle

JUAN E. SORNITO, *Centra Philippine University, Iloilo City, Philippines.*

It is impossible to construct a square equal to a given circle by the exclusive use of a straight edge and compass. This paper is intended to present a solution to this problem with a slight departure from the purely "straight-edge-and-compass" method.

IF A SQUARE is to have equal area with a given circle, then $s^2=\pi r^2$ or $s=\sqrt{\pi}r$, where r is the radius of the given circle and s is the side of the required square. Therefore s is the magnitude to be determined. The first step to determine s is to represent π by a straight line. This line is equal to the circumference of a circle of unit diameter. Hence, it is necessary to "rectify" a circle whose diameter is equal to 1, that is, to determine a straight line whose length is equal to the circumference of a circle of unit diameter. The second step is to represent by a straight line the square root of π. The third step is to represent s, the product of π and the radius of the given circle. The fourth and last step is to construct a square with s as a side. These steps are taken successively in the following solution.

SOLUTION

Given: The circle with center at 0 and diameter equal to 1.

Problem: To construct a square equal to the given circle.

Construction: Let AY be perpendicular to AX (see figure). The circumference of the given circle is equal to $2\pi r$.

Step 1.

(a) Measure AP equal to $\frac{1}{4}r$.
With P as center, construct a major arc AB equal to $\frac{3}{4}$ of the circumference. (Construct angle APB equal to 90°) Then major arc AB is equal to the circumference of the given circle.

$2\pi(\frac{1}{4}r)(\frac{3}{4})=2\pi r$ length of the arc AB.

(b) Measure AQ equal to $\frac{3}{2}r$.
With Q as center, construct major arc AC equal to $\frac{2}{3}$ of the circumference. (Construct angle AQC equal to 120°). Then arc AC is equal to the circumference of the given circle.

$2\pi(\frac{3}{2}r)(\frac{2}{3})=2\pi r$ length of the arc AC.

(c) With R $(AR=2r)$ as center, construct arc AD equal to the semicircumference.
This is obviously equal to the circumference of the given circle, since the radius is equal to $2r$.

(d) Measure AS equal to $3r$.
With S as center, construct arc AE equal to $\frac{1}{3}$ of the circumference. (Construct angle ASE equal to 120°.) Arc AE is equal to the circumference of the given circle.

$2\pi(3r)(\frac{1}{3})=2\pi r$ length of arc AE.

(e) $AD=4r$.
With D as center, construct arc AF equal to $\frac{1}{4}$ of the circumference. (Construct angle ADF equal to 90°.) Arc AF is equal to the circumference of the given circle.

$2\pi(4r)(\frac{1}{4})=2\pi r$ length of arc AF.

(f) Measure AT equal to $6r$.
With T as center, construct arc AG equal to $\frac{1}{6}$ of the circumference.

(Construct angle ATG equal to 60°.) Arc AG is equal to the circumference of the given circle.

$2\pi(6r)(\frac{1}{6}) = 2\pi r$ length of arc AG.

(g) Measure AU equal to $8r$.
With U as center, construct arc AH equal to $\frac{1}{8}$ of the circumference. (Construct angle AUH equal to 45°.) Arc AH is equal to the circumference of the given circle.

By continuing the process, taking one longer radius at a time, the arc will approach the straight line AX. Connect the points $A, B, C, D, E, F, \cdots$, etc., with a smooth curve and prolonged until it crosses the line AX at I. It will be observed that the arc approaches AI, as the radius approaches ∞. Therefore AI is equal to the circumference of the given circle, equal to π.

Step 2.

To determine the square root of π or AI.
Measure KI = the diameter of the given circle. Upon KA as diameter, construct a semicircumference. At I, erect a perpendicular to KA. Let this perpendicular intersect the semicircumference at J. Then IJ is equal to the square root of π, since IJ is the mean proportional between KI (1) and AI (π).

Step 3.

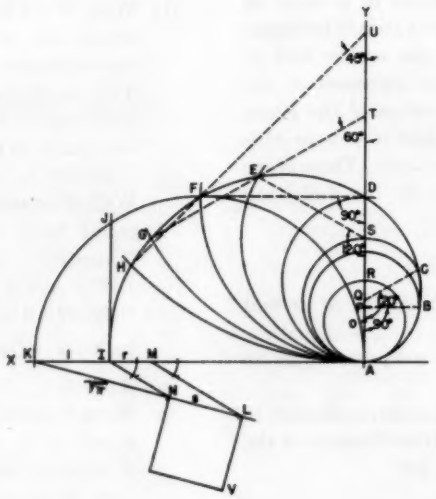
To determine s or $\sqrt{\pi}r$.

From K draw KL making any convenient angle with KA. Measure IM equal to r, and KN equal to IJ. Draw NI and ML parallel to NI. Then

$$\frac{KI}{KN} = \frac{IM}{NL} \text{ or } \frac{1}{\sqrt{\pi}} = \frac{r}{s} \text{ or } s = \sqrt{\pi}r$$

Step 4.

With s as the length of a side, construct a square. This square has the same area as the given circle, i.e. $s^2 = \pi r^2$. The square NV is the required square.



In the figure:

$AO = r$	$AS = SE = 3r$	$AI = \pi$
$AP = PB = \frac{4}{4}r$	$AD = DF = 4r$	$KI = 1$
$AQ = QC = \frac{3}{2}r$	$AT = TG = 6r$	$IJ = KN = \sqrt{\pi}$
$AR = RD = 2r$	$AU = UH = 8r$	$IM = r$

$NL = s$ the side of the required square

Another approach to the nine-point circle

JOHN SATTERLY, *University of Toronto, Toronto, Canada.*

An old friend in new dress.

1. Let ABC (Fig. 1) be a plane $\triangle$, and D, E, F the mid-points of its sides. The $\triangle DEF$ is similar to the $\triangle ABC$ and of one-half its linear dimensions. Let O and N be the centers of the circumcircles of the $\triangle ABC$ and DEF. (The diagram shows the usual method of their location.) Evidently the radius of the circumcircle of the $\triangle DEF$ is one-half the radius of the circumcircle of the $\triangle ABC$.

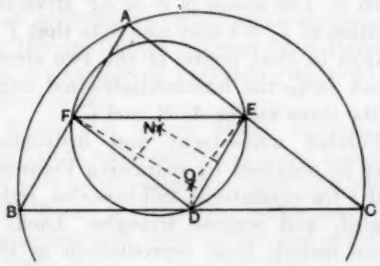


Figure 1

2. Draw the perpendiculars AK, BL, and CM (Fig. 2) from the vertices of the $\triangle ABC$ upon the opposite sides. These perpendiculars are called the altitudes, or orthogonals, of the $\triangle ABC$—I prefer the latter term. These orthogonals are concurrent at H, the orthocenter of the $\triangle ABC$. Bisect HA, HB, and HC at P, Q, and R respectively. Complete the $\triangle PQR$. Evidently the $\triangle PQR$ is similar to the $\triangle ABC$ and of one-half its linear dimensions. The diagram shows the construction for its circumcenter N'. Obviously the radius of the circumcircle of the $\triangle PQR$ is one-half the radius of the circumcircle of the $\triangle ABC$.

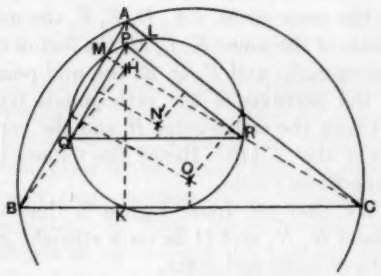


Figure 2

3. Figure 3 shows the two $\triangle DEF$ and PQR superposed on the same $\triangle ABC$. It also shows that the N of Figure 1 is the same point as the N' of Figure 2 and that the two $\triangle DEF$ and PQR have the same circumcircle.

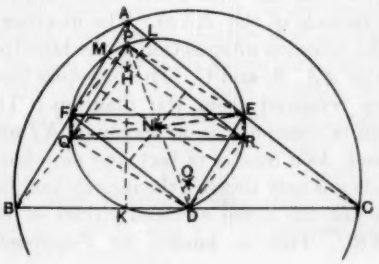


Figure 3

For proof, join FQ and ER to get a rectangle $EFQR$ whose diagonals intersect at the middle point which is obviously N (or N', which we shall call in future N). Also join DQ and EP and to get another rectangle $DEPQ$ whose diagonals also

intersect at N. Join DR and PF for a third rectangle, $DRPF$, whose diagonals intersect at N. Thus we conclude that the $\triangle DEF$ and PQR have a common circumcircle and that DP, EQ, and FR are diameters of this circle.

Also since each of the $\triangle DKP$, ELQ, and FMR is a right triangle the circumcircle of the $\triangle DEF$ and PQR passes through K, L and M, the feet of the orthogonals.

We have, therefore, nine points all lying on the same circle, viz., D, E, F, the mid-points of the sides; K, L, M, the feet of the orthogonals; and P, Q, R, the mid-points of the portions of the orthogonals lying between the orthocenter H and the vertices of the $\triangle ABC$. Hence the name—the *Nine-Point Circle*.

We also see from Figure 3 that the points H, N, and O lie on a straight line with N as its mid-point.

4. Figure 4 shows also that the centroid G (obtained as the common intersection of the three medians AD, BE, and CF) also lies on the same line and divides it so that $HG = 2GO$. This important line, $HNGO$, is called the *Euler Line*, after its reputed discoverer.

Another important property of the nine-point circle is illustrated in Figure 4. It shows that the nine-point circle touches the incircle of the $\triangle ABC$. The incenter I is the common intersection of the bisectors of the $\angle A$, B, and C. (The bisectors have been removed from the diagram.) The point of contact T is obviously on NI produced. As a matter of fact, the nine-point circle not only touches the incircle but also touches the three escribed circles of the $\triangle ABC$. This is known as *Feuerbach's Theorem*, after its first enunciator.

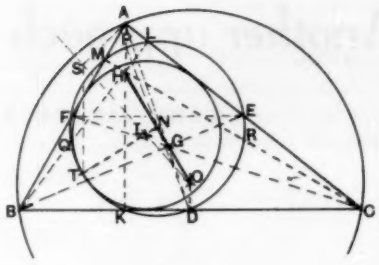


Figure 4

All the above results can, of course, be obtained theoretically by geometry and trigonometry as well as the fact that the distance NI is equal to the difference between the radii of the nine-point circle and the incircle. The position of T may, of course, be calculated. One method of locating T is shown in Figure 4. This is to join OI and drop a perpendicular AS upon it. The image of S in EF gives the position of T. We may also note that T is always in that region of the two circles which faces the intermediate-sized angle of the three angles A, B, and C.

Further amusement and instruction may be obtained by redrawing Figures 3 and 4 for equilateral, tall-isosceles, right-angled, and scalene triangles. Lack of space forbids their reproduction in this paper, but interest in them will certainly provoke the reader to get out his drawing instruments.

Further reference may be made to R. A. Johnson's *Modern Geometry* published by Houghton Mifflin Company, to other advanced textbooks on geometry and trigonometry, and to the mathematical magazines.

College mathematical training for students specializing in agriculture[1]

W. I. LAYTON, *Stephen F. Austin State College, Nacogdoches, Texas.*

A summary of a research study on the mathematical requirements of seventy-five agricultural colleges located in forty states.

OF INTEREST as we consider the topic before us is the doctoral dissertation by Clarence Monk, *The Place of Mathematics in Modern Agricultural Education in the United States.*[2] Monk corresponded with the heads of eight divisions of agriculture in one land-grant college in each of the forty-eight states and studied commonly used agricultural textbooks recommended by the land-grant colleges to determine the mathematical concepts employed in these books.

While the Monk study emphasizes findings in forty-eight land-grant colleges, the present investigation deals with opinions from seventy-five institutions of higher learning, both public and private, in forty states. It was felt that it might be of considerable interest to make a different type of analysis from that of Monk since there are quite a few colleges other than land-grant colleges offering programs of specialization in agriculture.

The purpose of the present study is to survey the mathematical requirements of colleges throughout the nation offering programs in agriculture and to analyze the opinions of representatives of these colleges as to the mathematical training most desirable for college students of agriculture.

In order to collect data a questionnaire was sent in November, 1954, to a wide distribution of colleges to be completed by the head of the department of agriculture or some member of his department designated by him. Requirements for students of agricultural engineering were excluded from the questionnaire, but requirements for those preparing for Smith-Hughes work in agriculture and all others who major or minor in agriculture were included.

The questionnaire was divided into four sections: (1) general questions concerning the individuals and institutions responding; (2) entrance requirements in mathematics; (3) present requirements in college mathematics; and (4) recommendations concerning college mathematics.

In each of the forty states participating in the survey there were usually from one to three institutions who responded. Enrollment figures of the colleges in the study show that thirty-three have fewer than 2500 students, seventeen between 2500 and 5000, and twenty schools have over 5000 students.

Let us consider now the entrance requirements in mathematics. The question was asked, "How many high school units in mathematics are required for entrance for your freshman agricultural students? (Do not include agricultural engineering students.)" The seventy-five colleges answering this question have requirements for freshman agricultural students ranging

[1] Condensed version of a paper presented at the Fifteenth Summer Meeting of The National Council of Teachers of Mathematics at Bloomington, Indiana, August 21–24, 1955.

[2] (New York: Columbia University, 1952).

from zero to three high school units in mathematics for entrance. Thirty-five schools require at least one unit of high school algebra while twenty-six require a unit of plane geometry for entrance. Only one of the colleges specifies general mathematics. Some schools require entrance units in mathematics but do not name specific courses.

The third division of the questionnaire concerned present requirements in college mathematics. The first question in this division asked what mathematics is required in the Smith-Hughes program for prospective agriculture teachers. In the schools which offer Smith-Hughes work, the courses most frequently required are college algebra by twenty schools, agricultural mathematics by ten, and trigonometry by seven. Eleven of the colleges require no mathematics in the Smith-Hughes program. For the schools which require mathematics here, the mean of all mathematics required is 3.91 semester hours.

The second question in this division of the questionnaire asked what mathematics is required of students who major or minor in agriculture other than those in Smith-Hughes training or agricultural engineering. Thirty-three colleges which offer majors or minors in agriculture require college algebra. Nineteen require trigonometry, and twelve agricultural mathematics. No mathematics is required by twelve of the colleges. However, in the colleges which do require mathematics, the mean is 5.55 semester hours.

The fourth section of the questionnaire dealt with recommendations in college mathematics. The first question was: "What mathematics do you think should be required in the Smith-Hughes program for prospective agriculture teachers?" Thirty-two colleges recommend agricultural mathematics in the Smith-Hughes program. Twenty-six specify college algebra and eighteen trigonometry. The mean of mathematics recommended for the Smith-Hughes program is 4.92 semester hours.

The second question in the fourth part of the questionnaire dealt with the mathematics which should be required of students who major or minor in agriculture other than those in Smith-Hughes training or agricultural engineering. College algebra and agricultural mathematics are both recommended by thirty-three schools and trigonometry by twenty-three schools. This is in contrast to the present requirements in the colleges studied where almost three times as many colleges require college algebra as require agricultural mathematics. The mean of all mathematics recommended for students who major or minor in agriculture is 5.43 semester hours.

The last question in the final division of the questionnaire was: "If you favor agricultural mathematics instead of college algebra and/or trigonometry, will you please react to the following topics which might be included in agricultural mathematics." Seventy-three items were included with this question and all respondents who favored agricultural mathematics were asked to vote whether each topic was highly desirable, desirable, or should be omitted from a course or courses in agricultural mathematics. Those answering the questionnaire were also given the opportunity to indicate any other topics which they felt were important. Forty-six topics were declared highly desirable in agricultural mathematics by fifty per cent or more of the forty colleges included in this study that advocate agricultural mathematics. These forty-six topics in the main could be classified as arithmetic, algebra, and their special applications to agricultural problems. There were also a few topics which are frequently treated in business mathematics, physics, statistics, and trigonometry.

SUMMARY

1. Requirements for freshman agricultural students ranged from zero to three high school units in mathematics for entrance. Algebra and plane geometry were the courses most frequently specified.

2. Of the mathematics required for Smith-Hughes programs, college algebra was the subject required most frequently.

3. College algebra led again in the mathematics required for those who major or minor in agriculture. This did not include Smith-Hughes students or agricultural engineers.

4. Under the heading of mathematics recommended for those specializing in Smith-Hughes training, agricultural mathematics was in first place.

5. College algebra and agricultural mathematics tied for top place in the list of mathematics courses recommended for majors and minors in agriculture other than those in Smith-Hughes training or agricultural engineering.

6. The recommendations resulting from this study are quite different from the requirements now made by the colleges. The data seem to indicate that there is a growing interest in requiring agricultural mathematics instead of college algebra and/or trigonometry in the curriculum for college students specializing in agriculture

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The visible figures by which principles are illustrated should, so far as possible, have no accessories. They should be magnitudes pure and simple, so that the thought of the pupil may not be distracted and that he may know what feature of the thing represented he is to pay attention to.

Thus, when the pupil comprehends clearly, by means of dots arranged in a rectangle, that three fives contain the same number of units as five threes, . . . then it may be expressed to him in general form, $a \times b = b \times a$.—*Committee of Ten*, 1892

The progress of mathematics is not smooth, nor is the science as the layman imagines, a collection of subtle principles and infallible results, springing mysteriously yet convincingly into the minds of their inventors. Its discoveries have, in general, not won immediate or universal acceptance, for mathematics, like every other system of organized knowledge, owes its development to the insight of thinkers whose creative imagination has led them to results which often startled themselves and their contemporaries.—*M. Black*, The Nature of Mathematics.

A simple but fruitful arithmetic

RICHARD V. ANDREE, *University of Oklahoma, Norman, Oklahoma.*

Teachers who are looking for enrichment material will find that these ideas supply an excellent introduction to some worth-while ideas.

BEING MATHEMATICS TEACHERS, you of course know all about arithmetic. Modern computing machines can perform two thousand multiplications or divisions of ten digit numbers in, not a week, or a day, or an hour, not even in a minute, but in one second! To be truthful, such big numbers frighten me. I'm going to talk about a system having only *seven numbers*, 0, 1, 2, 3, 4, 5, 6.

What can you do with only seven numbers? You can do almost anything you can do in ordinary arithmetic. The secret is that if we ever arrive at a number other than 0, 1, 2, 3, 4, 5, 6, we simply subtract (or add) a multiple of 7 which will give us a number in our system. Saying it in another way, two numbers are equal, mod 7, if they yield the same remainder when divided by seven. For example, $a=b$ mod 7 if $a=b+k\cdot 7$ for some whole number k. To remind you that we are using special rules I'll write $\equiv$ for equality in place of the usual $=$.

In this system $3+4\equiv 0$ and $3-4\equiv 6$.

You can see that negative numbers are unnecessary. Also $3\times 4\equiv 5$ but what about $3\div 4=?$

Let us examine exactly what we mean by the fraction $3\div 4$. It is the unique solution of the equation $4x=3$. Similarly we define the mod 7 solution of $4x\equiv 3$ to be $3\div 4$ in our new system.

$$4x\equiv 3$$
$$x\equiv 6 \text{ mod } 7.$$

Thus $3\div 4\equiv 6$. (Try it and see, $4\cdot 6=24\equiv 3$, mod 7.) Isn't this an interesting system? It permits us to do division without having to use fractions—indeed the system contains no fractions!

Let us examine the quadratic equation $x^2+x+5\equiv 0$. Since there are only seven numbers in this system, we may solve the equation by direct substitution, finding that $x\equiv 1$ and $x\equiv 5$ are solutions. (Note that they are *not* imaginary numbers.) On the other hand, the quadratic equation $x^2+2x+5=0$ has no solutions whatsoever in the mod 7 system.

By studying this very simple system you may gain great insight into the heart of arithmetic and algebra. Even more can be gained by also studying the mod 6 system. Mod 6 system contains only six numbers 0, 1, 2, 3, 4, 5 and $a\equiv b$ mod 6 if $a=b+k6$ for some whole number k. The mod 7 system obeys the usual laws of algebra but the mod 6 system violates the cardinal rule by which algebraic equations are solved, namely $AB=0$ *if and only if* $A=0$ or $B=0$. Note that $4\neq 0$, and $3\neq 0$, but $4\times 3=12\equiv 0$ mod 6. The quadratic equation $2x^2+4=0$ has solutions $x=1$, 2, 4, 5 (Bargain day!) while $x^2+2=0$ has only $x=2$ and 4 as solutions. Thus "dividing through the equation" has lost two roots for us. Why? I trust you will carry out further experiments in these interesting modern number systems. Your students may enjoy them while discovering some fundamental concepts behind the algebraic and arithmetic manipulation which they so glibly perform.

• HISTORICALLY SPEAKING,—

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

The history of mathematics as a teaching tool

Raleigh Schorling once commented that for all that a knowledge of the history of mathematics had long been advocated by himself and others as a necessary part of a mathematics teacher's kit of tools, it still seemed to have had little effect on actual classroom practices. The purpose of this note is to suggest principles and procedures for making history a more useful and effective teaching tool. Additional principles relating to the use of historical materials in a classroom, concrete illustrations of these new principles or additional illustrations of those set forth in this article, pictures or descriptions of historically based or related projects, lists of historical materials and books which may be useful will all be welcomed by this department of THE MATHEMATICS TEACHER.

There are at least three broad categories of functions which may be performed by properly used historical materials. These are: (1) they may clarify meanings, give insights, and sharpen understandings of mathematics itself; (2) they may give students desirable "appreciations"; and (3) in addition to contributing directly to the achieving of such desired outcomes as mathematical understandings and appreciations, they may also serve as primarily a pedagogical device for improving instruction, that is as a methodological tool.

One of the more important understandings to be sought in teaching mathematics is an understanding of its structure, which involves in turn an understanding of the nature and role of undefined terms, definitions, axioms, and the logical deduction of theorems. For example, in geometry a presentation and discussion of Euclid's definitions of a *point* ("a point is that which has no part") and a *line* ("a line is breadthless length")[1] help to bring out the need for undefined terms, and illustrate that intuitive and physical concepts which are *not* mathematics have nevertheless often stimulated the thinking of mathematicians and influenced their choice of undefined terms and axioms. This sort of discussion also helps students understand the "ideal" nature of true points and lines as opposed to the physical representations of points and lines which they draw. Both the utility and the dangers of diagrams and thinking based on them are clearer for a discussion of early ideas and how they have changed.

History helps in an even more concrete way with many terms which youngsters must come to understand. For example, the stories of the words *sine, zero, root, radical,*[2] and many others will help students understand, remember, and recall them.

A student can hardly fail to understand and retain the meaning of *isosceles* after he has heard an analysis of the word *isosceles* into its Greek forbears *isos,* equal, and

[1] Sir T. L. Heath, *The Thirteen Books of Euclid's Elements* (Cambridge: The University Press, 1908), I, 154.

[2] Some of these are given and methods of finding others will be found in P. S. Jones, "Word Origins," THE MATHEMATICS TEACHER, XLVII (March 1954), 195 ff.

Figure 1

skelos, legs, and has participated in a listing of other "iso" words in English (isocline, isotherm, isobar, etc.), and has discovered that *skeleton* was the Greek word for a dried body. If the student is also shown a cardboard or wooden model of the Egyptian leveling device sketched in Figure 1, he will not only be further helped to remember what an isosceles triangle is, but he also can *discover* for himself (led by some questioning if necessary) such properties of the isosceles triangle as that its altitude bisects the base and that, conversely, the perpendicular bisector of its base passes through its vertex.

The story of the development of non-Euclidean geometry should be told, from early attempts to find a "better" axiom to replace Euclid's fifth or parallel postulate, to attempts to prove the postulate, to the essentially independent and simultaneous development of hyperbolic geometry by the Russian Lobachewsky and the Hungarian Bolyai. Not only does this story clarify the nature and role of axioms in modern mathematics, but it simultaneously teaches a number of things which we will list here but classify under our second heading, "appreciations."

These are:

1. The continuity of the growth of mathematics which often leads to simultaneous discoveries (the German Gauss also had essentially the ideas of Lobachewsky and Bolyai)

2. The internationalism of mathematics

3. The operation of intellectual curiosity and aesthetic sensitivity as motives and stimuli for developing mathematical theory (None of the developers anticipated any practical use for this geometry and much of the earlier dissatisfaction with Euclid's postulate was based on such aesthetic considerations as its complicatedness, its nonself-evident nature, and the fact that its converse is a theorem.)

4. Often totally unanticipated applications come after the development of a mathematical theory, as when a non-Euclidean geometry was found useful in the mathematical formulation of a relativity theory (which latter, in a sense, led to the development of atomic energy).

5. The rapid current growth of both pure and applied mathematics and the relationships between them. (These are shown by the fact that Riemann's elliptic non-Euclidean geometry was presented in 1854, about twenty years after Bolyai's and Lobachewsky's work, and that the tensor analysis by which the geometric ideas important in relativity are most easily developed was invented by the Italian, Ricci, prior to 1890 and has been extensively applied since 1920 in such areas as the theory of elasticity as well as in geometry and relativity.

Much of this material has to be presented by mere talk, it is true, but there are some expositions of non-Euclidean geometry which can be used with secondary school pupils.[3]

This should help to convince pupils that

[3] See Lillian R. Lieber, *Non-Euclidean Geometry; or, Three Moons in Mathesis* (Lancaster: The Science Press, 1940); or the appropriate chapter in Moses Richardson, *Fundamentals of Mathematics* (New York: Macmillan, 1941), which gives a sequence of simple theorems, or in H. R. Cooley, D. Gans, M. Kline, H. E. Wahlert, *Introduction to Mathematics* (Boston: Houghton Mifflin Co., 1949) or in Richard Courant and Herbert Robbins, *What is Mathematics?* (New York: Oxford, 1941). The recent reprint of Roberto Bonola, *Non-Euclidean Geometry* (New York: Dover Publications, 1955), contains translations of an historical account and of some of the original works, while Harold E. Wolfe, *Introduction to Non-Euclidean Geometry* (New York: The Dryden Press, 1945), approaches it in a manner similar to that of high school geometry.

rigorous proofs and careful criticisms of axioms have produced new and useful mathematics and will do so again. A further example of this is to be found in the development of such modern algebraic theories as quaternions and algebraic numbers. This followed Hamilton's recognition of the importance of the associative and other axioms of elementary algebra in his formulation of complex numbers as "number couples."[4]

Quaternions are not altogether beyond superior secondary school students, but so-called "modular arithmetics" and scales of notation are topics which may be more easily studied at the secondary level and which will also help students understand the motives of mathematicians, and the role of axioms and rigor in extending mathematics.

The history of the binary and sexagesimal systems shows the intertwining of intellectual curiosity and application as motives for advancement. It also shows the interrelationships between cultures as a whole and mathematics.

For example, the development of agriculture, division of labor, and trade by the Babylonians was made possible by fertile, watered soils in a warm climate. Agriculture produced an interest in both the calendar and astonomy. This grew along with the arithmetic and mensuration of commerce and construction. Babylonian sexagesimal fractions were later taken over by Greek astronomers and were passed by them to the succeeding Arabic civilization. In translations from Arabic into the scholarly Latin of medieval western Europe, sixtieths and sixtieths of sixtieths became the "pars minuta prima" and "pars minuta secunda" from which we got our "minutes" and "seconds."

Although more recently devised units of angular measure were defined entirely arbitrarily from a strictly logical viewpoint, they were consciously and deliberately devised for greater simplicity and convenience as opposed to the purely historical and almost random reasons behind our present system of degrees, minutes and seconds. In this particular case, *only* history can give the real reasons and a complete understanding of the system. The history of other angular units not only makes them more meaningful but also sheds light on the nature, source, and role of definitions in mathematics.[5]

In the course of presenting illustrations of my first contention that the history of mathematics may be used to teach some deeper understanding of the structure of mathematics, the roles of rigor and intuition, and of the mathematical motivation of some topics, I have incidentally given examples of my second point. This is that the history of mathematics may be used to teach several "appreciations." Among these are: (1) an appreciation of the abstract and logical nature of mathematics; (2) an appreciation of how the abstract systems of mathematics may be associated with physical systems and then used to solve "applied" problems; (3) a perception and appreciation of interrelationships within mathematics as well as with other areas of abstract thought such as philosophy, logic, and religion, and with the culture as a whole; (4) an insight into the motives which lead people to do mathematics; (5) an appreciation of the continuity and vigor of the growth of mathematics up to the present day; (6) a realization of the international nature of mathematical interests and productivity.

The pedagogical utility of the history of mathematics embraces all that has been claimed for it above. In addition, it may serve the more purely pedagogical purpose of: (1) creating interest of the type which

[4] P. S. Jones, "Complex Numbers: An Example of Recurring Themes in the Development of Mathematics, II," THE MATHEMATICS TEACHER, XLVII, (April 1954) 260. Reprints of three articles in this series may be purchased from the National Council of Teachers of Mathematics.

[5] P. S. Jones, "Angular Measure—Enough of Its History to Improve Its Teaching." THE MATHEMATICS TEACHER, XLVI (October 1953), 419 ff.

motivates; (2) providing enrichment materials which may be used for differentiated assignments, projects, and the construction of teaching aids; (3) suggesting devices for introducing topics and for aiding students to "discover" new concepts; (4) assisting both the teacher and the student to achieve and maintain a perspective on both the difficulty and the importance of comprehending the abstract nature of mathematics.

When a student hears how long it took for the concept of negative numbers to develop and become accepted, used, and understood, he does not feel quite so concerned that the concept didn't come to him easily. In fact these numbers which everyone today sees all around him on thermometers, stock market reports, antifreeze charts, etc., were called "fausse" or false by the great mathematician-philosopher René Descartes. But the fact that Descartes went ahead and used these numbers suggests to students and teachers that this concept was not an easy one to develop, that we have made really tremendous growth, and that what is today new, abstract, and difficult may in a not-too-distant future be relatively simple and common. This will happen, however, only as the concept is worked with, represented in various ways, and applied.

The word "imaginary" is a misnomer also used by Descartes which has unfortunately stayed with us to the present day. The story of the earlier beginning of imaginaries in Cardan's famous problem may be a good way to introduce the study of complex numbers. Are there two numbers whose sum is ten and whose product is forty? Cardan said that this problem was "obviously impossible." I don't know on what basis he called it impossible, but here is a good way to challenge youngsters by asking them how Cardan might have drawn this conclusion.

Students can easily be led to suggest, if they do not readily think of them for themselves, such approaches to Cardan's problem as numerical experimentations and graphs. Numerically, by taking such products as 1×9, 2×8, 3×7, 4×6, 5×5, 6×4, etc., one soon sees that 25 is the maximum product of two numbers whose sum is 10. This same result appears graphically if one writes and then plots the graph of the equation $y=x(10-x)$. Here, too, one can emphasize both the importance of the *if* in a mathematical statement and the roles of intellectual curiosity and analogy in mathematical discovery. Cardan's problem is impossible of solution *if* one limits oneself to the field of real numbers.

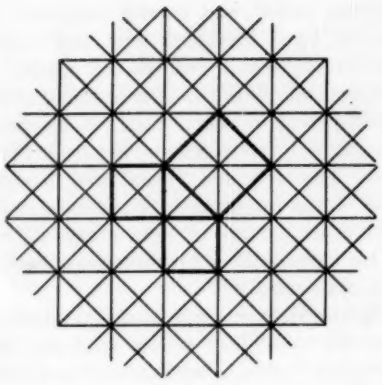


Figure 2

Cardan, motivated by a curiosity about what might happen in this "impossible" situation if he followed the same process which he used in possible cases, applied the method of completing the square to $x(10-x)=40$ and came up with $x=5\pm\sqrt{-15}$. He then showed that if $\sqrt{-15}$ obeyed rules analogous to those followed by ordinary radicals, the equation was satisfied.[6]

Historical facts and speculations furnish many other suggestions for leading students to "discover" mathematical ideas. For example, some persons have thought that the Pythagorean theorem might have been first discovered in the case of the isosceles right triangle where it is geometrically quite obvious and might have been visualized in a tile floor or wall, as shown in Figure 2.

[6] P. S. Jones, "Complex Numbers," *op. cit.*

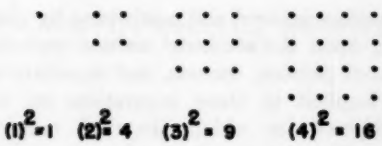


Figure 3

This is speculation, but it seems quite certain that the discovery of one formula for Pythagorean triples of numbers grew out of Greek interest in "figurate" numbers—numbers represented by geometric figures composed of dots. Thus Figure 3 shows the first four square numbers, and Figure 4 shows the first four triangular numbers. If we consider Figure 5 to represent the nth square number, we can then see that the next square number can be formed by adding n dots on each of two sides and filling in another dot at their intersection.

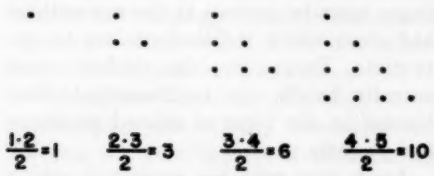


Figure 4

In other words $(2n+1)$ dots added to the nth square number makes it into the $(n+1)$th square number, or $n^2+(2n+1) = (n+1)^2$. This is almost a trivial identity when written this way, but students not only may be led to see in it a new interpretation of the square of the binomial $(n+1)$, but they may also be led to discover the numerical interpretation suggested by Table 1. The sum of the first n odd numbers is nth perfect square.

This ancient result can be made into a useful device for computing square roots on a modern desk calculator, and leads to the converse that if $y=f(x)$ is a quadratic function represented by a table in which the xs appear at equal intervals, then the second differences of the y values will be a constant. This is a special case of a fundamental theorem in the calculus of finite differences developed by Brook Taylor and Leonard Euler in the eighteenth century and useful to actuaries, statisticians, and engineers or to anyone who may wish to fit curves to data or do accurate interpolation or numerical integration.

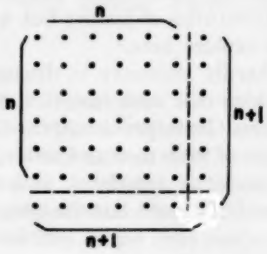


Figure 5

This one example not only shows chronologically the development and extension of a concept, but also displays interrelationships between geometry, number theory, algebra, analytic geometry, and both the finite and infinitesimal calculus as well as interrelationships between mathematics and its applications.

Further, a youngster who, perhaps, with the teacher's help, sees that $n^2+(2n+1)=(n+1)^2$ is "almost" the Pythagorean theorem might then be led to try what would happen if $(2n+1)$ were a per-

TABLE 1

n	Odd Numbers	Sum of n Odd Numbers
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
.	.	.
.	.	.
.	.	.

fect square, say m^2. He should come out with

$$\left(\frac{m^2-1}{2}\right)^2+m^2=\left(\frac{m^2+1}{2}\right)^2,$$

which will manufacture a set of "Pythagorean triples" whenever m is an odd number. If he were then to try to get rid of this last restriction and further seek a formula which would give him *all* Pythagorean triples he would be discovering for himself a fascinating trail which has wandered through centuries of history but which we can not explore here.[7]

It is hardly necessary to discuss in detail the idea that such historical topics as the famous Greek problems; the fascinating stories of such men as Cardan, Galois, Gauss, and the anecdotes about Archimedes and Fermat's last theorem all furnish excellent club topics and interesting enrichment projects.

Many models, some of real value as teaching aids (e.g., the abacus, Napier's bones, trisectors, Gunter's scale) are also suggested by the history of mathematics as either student or teacher projects which make interesting demonstrations and bulletin board material.

Even a brief mention in class of who did it, why he did it, or what was the first or last, biggest or best solution to a problem develops interest and motivation by playing upon the students' natural curiosity about persons, reasons, and superlatives.

Implicit in these suggestions on the purposes for which historical material may be used in teaching have been suggestions as to *how* they may be used; e.g., to lead to discovery, to enrich the course and to provide topics for projects.

But if the history of mathematics is to be a real teaching tool a few other guides must be followed. History is not mere names and dates, but ideas developing for some reason. Names and dates will soon bore more than they interest. The mathematical ideas must be pointed out clearly and new ideas illustrated concretely, explored, and observed from several viewpoints. This means that the teacher must be sure that the interrelationships referred to so often above are actually perceived by the student, i.e., the teacher must see to it that a whole story is told with diagrams and worked examples. A figurative finger must be pointed at the connections and ideas which the students are to appreciate. Frequently, the student must actually handle the mathematical ideas himself in the form of related problems and projects.

Again, this note has suggested only a few of the materials, sources of ideas, and topics to be used in making the history of mathematics a teaching tool. We invite readers to add ideas and concrete examples to this discussion. Send them to the editor!

[7] P. S. Jones, "Pythagorean Numbers," THE MATHEMATICS TEACHER, XLV (April 1952), 269 ff. gives a general formula and cites references wherein one may trace its development.

SIAM—the newest mathematical society

What is SIAM?

It is the Society for Industrial and Applied Mathematics, an organization of fairly recent origin, having been incorporated on April 30, 1952. Its aims and objectives are: (1) to further the application of mathematics to industry and science; (2) to provide basic research methods and techniques useful to industry and science; and (3) to provide mediums for the exchange of information and ideas between mathematicians and other scientific and technical personnel.

It seemed to those who conceived SIAM that a large gap existed in our professional structure

Continued on page 79

Edited by Lucien B. Kinney, Stanford University, and
Dan T. Dawson, Stanford University, Stanford, California

Removing the "dis" from dislike for mathematics

by Alice Hach, Racine Public Schools, Racine, Wisconsin

Ralph Pringle once wrote that "The junior high school age is the most unloved age but when we come to know it, it is the most lovable age." In view of this statement it seems important that we examine together this important three-year block in our educational training. We need to determine what can be done to develop well-adjusted individuals, with an appreciation for learning, with enthusiasm and curiosity for advanced work and study, and with ideals and a true sense of values.

We who are teaching junior high school mathematics are vitally concerned regarding the development of these strong personal traits in our students at the same time that we are building strong foundations in the field of mathematics. We realize that it is a waste of human resources whenever a student's personality or philosophy hinders him from making the utmost use of all the ability he possesses. We need to be concerned whenever a student chooses his course of study in the high school only on the basis of snap courses, avoiding advanced mathematics courses because these subjects take more time and because he dislikes mathematics.

This, therefore, is our task and our challenge—to remove the *dis* from the dislike for mathematics and create a liking for the subject. This does not mean that we are interested in developing watered-down junior high school mathematics courses or courses motivated merely by easiness or by games. We want mathematics to earn respect and appreciation through a thorough understanding of the subject and a recognition of its importance to the advancement and survival of a civilization.

What can we, as teachers, do to reach this goal? Three things are suggested for increasing the effectiveness of our mathematics program in the junior high school. First, it is important that anyone working in this area understands the true purpose and nature of the junior high school organization. Second, each teacher needs to know the emotional climate in a classroom which will result in the most effective learning. Third, each teacher needs to know techniques that help build this emotional climate in order that children may gain new insight in the study of mathematics and develop a liking, an appreciation, and a curiosity for the subject.

Let us first take a look at the junior high school organization. We must keep in mind that the junior high school is neither an enlarged elementary school nor is it a miniature senior high school. It is a school that exists for its own worth and its own function.

In Figure 1 we have an illustration of the typical elementary and the typical senior high school.

The elementary school generally consists of grades from kindergarten through

Figure 1

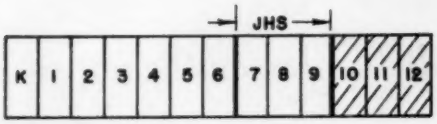


Figure 2

six, and the senior high school grades ten through twelve.

It is needless to say to teachers that the training, and the emphasis in this training, differ considerably for the elementary and the senior high schools. If we were to place a segment of the elementary school in the junior high block as shown in Figure 2, we would fail to prepare pupils for this change of emphasis before they reach the high school level. Nor are we taking advantage of the child's interest in new approaches at the time when he himself is beginning to make the change from childhood to adulthood and wants to be treated as grown up despite the fact that he still clings to many childish ways.

Likewise putting a segment of the senior high school in the junior high school block, as in Figure 3, is not the answer. This merely places the child in a more adult organization at a still earlier age and with the same lack of preparation for this adjustment.

Therefore, we can understand that neither elementary type teaching or senior high teaching is adequate or effective at this level. Junior high school teaching must be a combination of these two methods so that a child may move from the elementary program into the senior high school program almost as easily as he moved from the fifth to the sixth grade.

The diagram in Figure 4 by Dr. James Glass shows most effectively how a gradual transition can be made from the elementary school to the senior high school.

This is not merely a matter of organization but should be a part of every teacher's thinking as he works at the various levels in the junior high school. The seventh grade will use many of the methods and techniques used in the elementary grades. However, when it comes to the new emphases on marks, on homework, on secondary approaches to mathematics, definite time should be taken to help the child understand these new emphases. We must not assume the child automatically adjusts to the junior high school by simply being placed in a junior high school building.

The eighth grade is often the grade which teachers find difficult. This is an unpredictable age since childish traits and adult traits seem to be intermingled. Here we need enough of the elementary approach to stabilize the group, but enough of the secondary approach to challenge and hold the interest of the group. It is this balance that often makes the difference between an enthusiastic eighth grade class and a problem eighth grade.

As the child enters the ninth grade and is approaching the senior high school, much greater emphasis on the secondary level should be given. Grades, credits, and plans for the future, all take on new significance. Algebra becomes a mathe-

Figure 3

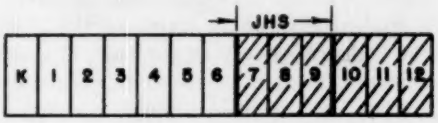
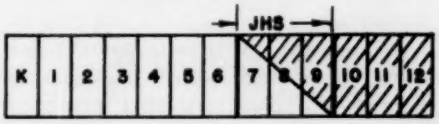


Figure 4



matics course for college entrance and general mathematics for business. This level is a popular level for the teacher since secondary methods tend to be more effective here.

As for the emotional climate which will result in the most effective learning, the seven needs listed by Dr. Louis Raths warrant our attention. They are:

1. The need for security
2. The need for success
3. The need for recognition
4. The need for sympathetic understanding
5. The need for acceptance, with respect of the child as he is
6. The need for status with peers
7. The need for increasing self-direction

It is important that a child feels secure with us and the classroom situation if he is to be free to learn. Yet many children at this age level feel quite insecure in the classroom. We teachers need to be aware of this and do all we can to build the feeling of security within the framework of the class.

Likewise, each child needs some form of real success. If it is impossible for him to secure success in the classroom, he will resort to any means to gain satisfaction, even if this success takes on the form of disrupting the well-planned routine of the classroom.

It is interesting that although the need for recognition is essential to learning, we often tend to neglect certain individuals. There is an interesting little device which will help us overcome this habit of neglect. It is to set a goal to recognize each individual in the class in some way and make a record each time a personal comment is made to the child. The comment might be a simple one, but it must be sincere. This device is recommended to you as challenging and interesting. It helps a teacher emphasize the positive rather than the negative in teaching.

As for the need for sympathetic understanding, there is perhaps no age at which a child needs more sympathetic understanding than in the junior high school. Often his parents do not understand him and as a result he becomes one of the unloved age, as Pringle explained. Do we as teachers really care when Johnnie gets a low grade, and do we let Johnnie know that we really care? Do we ask about an illness after an absence, or do we merely ask for the admittance slip? Stanley Hall once wrote, "There is no greater blessing that can come to a boy of this age when he does not understand himself than a good strong teacher who does understand him and has faith in him and will day-by-day lead him until he can walk alone."

It is also important that we accept with respect the child as he is. We may object to the thing a child is doing, but we should never object to the child himself. Perhaps there is no statement more effective when a discipline problem arises than to say with sincerity, "Fred, I like you very much, but I do not like what you are doing." The fact that you like Fred puts Fred in a frame of mind to accept criticism and to want to improve. The story of Bill illustrates this. The teacher said, "Bill, don't you know the difference between right and wrong?" Bill answered, "Sure I do." "But you always do the wrong thing," objected the teacher. Bill only grinned and said, "Well, that shows it ain't guesswork." Bill was not in the frame of mind to want to improve his ways.

The need for status with peers is so strong at the junior high school level that we should be cautious that we do not ridicule a child before the group. It is far better for us to build up the prestige of a junior high school child before the group and make personal comments privately. A child at this age cares far more what the class thinks of his actions than how a teacher feels. For example, if a teacher does not want a child in the class, the child often does not care. But if the class does not want him, he is deeply hurt.

The child at the junior high school level

needs increasing self-direction. We need to remember that learning takes place only if there has been a successful experience. Once the class has had success in self-direction, they can assume more responsibility. However, an unsuccessful experience indicates that the group was not ready for the amount of responsibility that had been given them. Self-direction should then be decreased and gradually rebuilt again. The child as well as the teacher needs to realize that this is a learning process and that often the process must be relearned at the junior high school level because the child himself has changed.

Once we understand the junior high school organization and know the emotional climate that we want in the classroom, we still have to know what techniques are effective to secure this wanted emotional climate.

For years a collection was made of children's remarks when evaluating their mathematics class in respect to things they liked about their class, things they did not like, and suggestions for changes. The children were made to feel that this information would be very helpful in improving the class for other boys and girls. Sincere opinions were urged. A few of the representative comments are being shared with you. No doubt these remarks could just as well have been made by children from any teacher's class. The significance of the remarks is that they were repeated enough times to indicate their importance to the junior high school boy and girl.

Appreciation for explanations and understandings:

The thing I like best in this class is the way you explain things. I feel I get more out of math when I know what it's all about and not just a bunch of numbers.

The things I have liked in this class are the ways you explain things so carefully. Most teachers run through the problems real fast and don't give a chance to the kids who don't understand it right away.

Appreciation for extra help versus required extra help:

I like the idea of being able to come in for advice, make up, etc. Your friendliness towards the class as a whole, and toward each of us was most encouraging.

By not requiring students to come in after school, you avoided the hard feelings that so often spring up between the teachers and the students.

Appreciation for recognition of individual differences:

I think it was very nice of you to help us in a lower class and I think if others need help you should have them in a lower class too.

Appreciation for assignments and homework in junior high school:

They say that after-school homework in high school is tremendous. I think this course has taught me to work harder at night and in that manner has better equipped me for heavier work in high school.

I particularly enjoyed the assignments. They were the right size and because of the size everyone did them with more enthusiasm.

Appreciation for self-evaluation:

The weekly tests are what I like because they give you a chance to know just how you are doing.

Appreciation for orderliness in the classroom:

I like the way you keep us controlled without getting crabby.

I have a suggestion that you always keep your great sense of humor that the kids like and they will work better with you that way than being crabby.

Appreciation for an interest in advanced mathematics:

I learned that I would like to continue in different types of math in high school.

I learned that math wasn't just numbers but a great deal of things. I feel now that there is an entire field and new phase opened to me.

Appreciation for learning to like mathematics:

Math has never been particularly easy for me but this year I found I could do it if I tried. This surprised me very much.

Math used to be my worst subject. Now it's my best and favorite subject.

To me this class seemed like the best class I ever had, although this is only my third year in America.

Appreciation for the appearance of the classroom:

I like the way you fix the room up. It makes it so pretty and more like home.

I like the colors and brightness in the room and the cheerfulness in it.

The general atmosphere in our class is very heartening when I'm feeling low.

Perhaps all of these remarks can be summed up with a comment written by a pupil to his teacher. It is a tribute to all teachers interested in boys and girls.

I feel that as long as there are people such as you, who care enough to correspond and encourage the youth of America like you have and are doing, there shall always be brotherhood and humanity in the United States.

What's new?

BOOKS

College

Mathematics of Business, Accounting, and Finance, 2nd ed., Kenneth Lewis Trefftzs and E. Justin Hills. New York, Harper and Brothers, 1956. Cloth, xii+591 pp., $4.50.

Topics in Number Theory, Vol. I, William Judson LeVeque. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956. Cloth, x+198 pp., $5.50.

Topics in Number Theory, Vol. II, William Judson LeVeque. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1956. Cloth, viii+270 pp., $6.50.

Miscellaneous

Spectroscopy at Radio and Microwave Frequencies, D. J. E. Ingram. New York, Philosophical Library, 1956. Cloth, xii+332 pp., $15.00.

BOOKLETS

A Description of the College Board Achievement Tests, College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, New Jersey, or Box 27896, Los Angeles 27, California. 135 pp., 50¢ per copy.

A Description of the College Board Scholastic Aptitude Test, College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, New Jersey, or Box 27896, Los Angeles 27, Calif. 63 pp., 50¢.

Bibliography of Mathematics for Secondary School Libraries, Department of School Services and Publications, Wesleyan University, Middletown, Connecticut. Bibliography of 100 books and list of periodicals prepared by Robert A. Rosenbaum and Louise J. Rosenbaum, free.

Offerings and Enrollments in Science and Mathematics in Public High Schools (Cat. No. FS 5.17:118), Superintendent of Documents, Government Printing Office, Washington 25, D. C., 24 pp., 15¢.

Original Investigation or How to Attack an Exercise in Geometry (10th Printing), Bonded Scale and Machine Company, 2176 South Third Street, Columbus 7, Ohio. Reprint of Monograph written in 1901 by Elisha S. Loomis, 63 pp., free.

Sharing the Risk—An Approach to Family Economic Security, Educational Division, Institute of Life Insurance, 488 Madison Avenue, New York 22, New York. 23 p. illustrated booklet with teacher's guide; classroom quantities of booklet and one guide, free.

FILMSTRIP

Learning New Numbers—Fractions, Filmstrip House, 347 Madison Avenue, New York 17, New York. Set of eight color filmstrips.

MAGAZINE

Age of Science (1957 Edition), Circulation Manager, Age of Science, Inc., 1968 Yale Station, New Haven, Connecticut. An annual guide to current developments and future opportunities in science and engineering; approximately 150 pp.; $1.00 plus 50¢ postage (check or money order should accompany request).

TABLES

Logarithm and Trigonometric Function Tables, On Card (No. 592), W. M. Welch Scientific Company, 1515 Sedgwick Street, Chicago 10, Illinois. Four-place values printed in black and red on stiff paper, 8½″×11″, $3.50 for package of 25.

TEST

Madden-Peak Arithmetic Computation Test, World Book Company, Yonkers-on-Hudson New York. Achievement test designed for high school students and prepared by Richard Madden and Philip Peak, with two equivalent forms and national norms; test booklets $3.00 per package of 35, net.

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

On popularizing mathematics...

When you enter the bookstall at a great American international airport and find Hogben's *The Wonderful World of Mathematics* stacked nonchalantly among books on travel, mountain-climbing and skin-diving, it must be admitted that mathematics has arrived. Along with music appreciation, popular mechanics, the scientific approach, popular astronomy, and the like, mathematics has finally been accepted by the celebrated man in the street. And this despite the fact that

> Mathematics is peculiar among all the sciences in that it does not lend itself in any way to popular exposition. There are no gadgets to display with a glib veneer of "explanation" on how they work. There are no experiments to perform magician-like in order to amuse or to entertain. Mathematics at any level requires individual careful thinking, which means hard work for most people and is downright distasteful to many. In order to understand and appreciate mathematics one has got to make up his mind to think. There is no facile alternative.[1]

We believe that Mr. Hacker's contention is essentially true. As Whitehead has said:

> Outsiders are apt to accuse our subject of being recondite. Let us grasp the nettle at once and frankly admit that in general opinion it is the very typical example of reconditeness. By this word I do not mean difficulty, but that the ideas involved are of highly specialized application, and rarely influence thought.[2]

Yet withal, there is eloquent testimony that it *can* be done: consider the pioneer writings of L. Hogben, E. T. Bell, Kasner and Newman, and the Liebers, as well as works of Tobias Dantzig, George Gamow, Edna Kramer, Alfred Hooper, Aaron Bakst, Morris Kline, D. E. Littlewood, Egmont Colerus, and others.

All the more reason, therefore, for calling our readers' attention to four more rather recent and eminently worthy claimants to that growing circle of books which are at once scholarly as well as popular expositions of a recondite subject. These are Karl Menninger's *Mathematik in deiner Welt*,[3] W. W. Sawyer's *Prelude to Mathematics*,[4] Rózsa Péter's *Das Spiel mit dem Unendlichen*,[5] and C. Stanley Ogilvy's *Through the Mathescope*.[6]

Menninger's book reminds one not a little of Kasner and Newman's classic, *Mathematics and the Imagination*, although it has much of the delightful flavor of the late Walter Lietzmann's writing. According to the author, the book was written for those who love mathematics and for those who hate it, and also for a third class of readers, those who say: "I like mathematics, but it doesn't like me." Readers who are familiar with Menninger's *Zahlwort und Ziffer* (a cultural history of the development of numbers) and his *Ali Baba und die 39 Kamele* can readily appreciate that rare gift of combining sprightly and scholarly writing. *Mathematik in deiner Welt* is really worth translating into English.

[1] Sidney Hacker, *Arithmetical Viewpoints* (1948), p. iii.

[2] A. N. Whitehead, *The Aims of Education* (1929, 1949), p. 84.

[3] (Göttingen: Vandenhoeck & Ruprecht, 1954), 233 pp.

[4] (Penguin Books, 1955), 214 pp.

[5] (Leipzig: B. G. Teubner Verlagsgesellschaft, 1955), 278 pp.

[6] (New York: Oxford University Press, 1956), 162 pp.

Those of you who enjoyed *Mathematician's Delight* have a pleasant surprise in store: in *Prelude to Mathematics*, Professor Sawyer has done it again. This time his lucid, pungent exposition deals not with the traditional mathematics of the scientist and the engineer, but with far more intriguing mathematics such as non-Euclidean geometries, algebra of logic, matrix algebra, determinants, projective geometry, modern geometry, transformations, homogeneous co-ordinates, groups, and the like. The exposition of these topics, considering the brief treatment, is unusually skillful. Perhaps the best part of this book is made up of the first five chapters which deal with the notions of mathematical beauty and power, the qualities of a mathematician, the pattern of mathematical thought, generalization in mathematics, and the unification of mathematics, that is, the interrelationships between various branches of mathematics.

The unique feature of Péter's book is the way in which a great variety of topics has been brought together with the aid of the concept of infinity. Just as the function concept can be made the unifying theme in algebra and elementary analysis, so here the notion of the infinite has been deftly made the leitmotif. In the author's own words:

> I love Mathematics not only because it is useful to Technology, but also because it is beautiful, because Mathematics can satisfy Man's puzzle instinct, and above all, because Mathematics can make the Infinite comprehensible.

Dealing as it does with somewhat more subtle and elusive ideas, the book calls for slightly more sophisticated readers; nevertheless, like the others, it is not a text nor a study book, but a *Lesebuch*—a popular rendition, eminently worthy of translation.

Our last contender in this popularity contest is also extremely fascinating and readable. It bears a striking resemblance to *Mathematics and the Imagination* (which seems to have set some sort of a pattern, and a very commendable one, although we hope it won't have too many imitators), Professor Ogilvy's book is slightly disappointing only because the sections are all too brief. The word *mathescope*, suggested by the late Professor Kasner, was invented by Watson Davis, the eminent popular writer of modern science.

Reform is in the air . . .

Yes, reform in mathematical education is very much in the air. The "new mathematics" is definitely on the way. To be sure, no one can know at this stage just what form it will ultimately take. It will be several years before it crystallizes. Meanwhile, all concerned will eagerly follow activities begun by the University of Illinois Committee on School Mathematics; the Mathematical Commission appointed by the College Entrance Examination Board at Princeton; the Educational Testing Bureau at Princeton; the Committee on the Undergraduate Program, Department of Mathematics, University of Kansas (Lawrence); the Special Committee of the National Council of Teachers of Mathematics; and other agencies and individuals concerned with curricular problems in secondary mathematics and in freshman college mathematics.

In view of the above, it may be of interest to note what J. Bronowski has to say, writing in a recent issue of *Science* under the heading *"The Educated Man in 1984."* In discussing the need for interpreting modern science to cultured, non-technically trained people, he says:[7]

> . . . I want to fix our minds much more firmly on the lively nonscientist. What can I propose for him?
>
> First, I want to propose less mathematics than he studies now, but of a more practical kind. That is, I want to change the stress from mathematical manipulation to the meaning of the sales tax and the twinning rate. The lan-

[7] *Science:* April 27, 1956, vol. 123, p. 712.

guage of mathematics is still taught as a dead language, nearly all grammar. I want instead to teach more translation, from the everyday facts into mathematics and thence back into the everyday.

As one corollary, I believe that we need to make statistical methods part of the education of everyone in schools and universities: and this for two reasons. One is that only from statistics can the nonscientist learn to use averages and approximations with confidence, to be unafraid of dividing the national debt or the German battle losses by the population, and to know in his bones the difference between a million and ten. The other is that modern mathematical statistics is a new view of science, which I believe will transform it and will replace the mechanisms of Newton by the more subtle concepts of modern physics.

There is another way in which I want to see the teaching of mathematics changed. I think we should be less preoccupied with number and quantity and more with relations of order and arrangement. This vision of the world by shape and structure is, I think, characteristic of the newer sciences, and the mathematics teacher can help to make it familiar.

In fairness to all concerned, it should be emphasized that the thinking at Urbana, Illinois and at Lawrence, Kansas is presumably concerned with our abler students, chiefly those who are going further in their study of mathematics and/or science, whereas Bronowski is more concerned with the contribution of mathematics to "general education."

Rounding numbers with a vengeance

It has become well-nigh universal to stress intelligent rounding of numbers in teaching arithmetic—and properly so. Recently we ran on to a practical application of rounding—via electronic computation—which we think our readers would like to know about. We allude to the "even dollar" pay plan put into operation by a well-known public utility company. In their own words:

Starting with the first week in June all employees paid on a weekly basis began receiving their pay under the new "even dollar" pay plan. This means that the pay envelopes contain bills only—no coins. Employees receive the amount due them rounded out to the next highest even dollar. The following week, the amount advanced the preceding week is deducted from the amount due that week and the roundout process is repeated.

This new procedure is made feasible by use of an electronic data processing machine recently installed.

A record of the prior week's advance is recorded on the magnetic tape upon which also are recorded all the other data pertaining to computations of the employee's pay. The machine then computes the net pay and prints the "even dollar" amount, together with other data, on the employee's pay statement. When the magnetic tape is run the next week, the machine takes account of the previous week's advance.

This procedure does away with the task of handling some 90,000 coins a week, weighing about half a ton, or some 20 pounds per paymaster's bag.

The wandering decimal point . . .

We cannot resist passing on the following little piece gleaned not so long ago from the *New York Times:*

A story last Sunday about American Telephone and Telegraph's record $2,100,000,000 expansion budget has caused quite a stir among mathematicians. It said erroneously that this meant an outlay of $66,408.62 every second. Several letter writers located correctly the misplaced decimal point—three digits to the left. The actual outlay is $66.40862 a second.

Quite a few mathematicians, however, forgot that 1956 is Leap Year. Hence, the company's spending goes on for 366 days. Their mistake puts them off only $5,737,704.77.

A column of unofficial comment.

How are we doing?

by H. Vernon Price, State University of Iowa, Iowa City, Iowa

The National Council of Teachers of Mathematics is dedicated to the task of improving the teaching of mathematics in this country. The goal is most worthy and one which will always be just out of reach, for nothing educational is so good that it can't be improved. To suggest, in a setting such as this, that today's schools and teachers are doing a pretty good job of training the youth of the land may be dangerous. Yet that is exactly what I propose to suggest.

In the last few years any number of articles have appeared—many within the covers of THE MATHEMATICS TEACHER—deploring the sorry teaching of mathematics which takes place in the secondary schools and the resulting adjustments which are forced on the colleges. The general impressions which these writings create are:

1. Teachers of high school mathematics are an ineffective, stupid lot.

2. The younger generation is going to the dogs.

3. The poor college teacher has been forced out of his ivory tower and plunged headlong into the world of reality.

4. The enemy is rapidly surpassing us in trained manpower, largely because our high-school-graduates-to-become-college-students know so little of the fundamentals of mathematics.

5. And IT'S ALL YOUR FAULT.

It is not the purpose of this article to try to prove or disprove these statements, nor to deny the validity of points 2, 3, and 4. (I assume that it is unnecessary to deny 1 and 5.) There are many spots in an educational hierarchy where improvements *can* and *should* be effected. One of these is undoubtedly in the recruiting and training of teachers, a dilemma whose resolution may be purely financial. In any event, we know that our best students shy away from secondary teaching and those who fill the gaps find it difficult to teach that which they do not know. It is not necessarily a matter of the blind leading the blind, but the situation does tend to propagate itself.

Unfortunately, this is only one of the many situations which offer opportunities for improvement. The over-all problem has many facets. Even so, I don't believe the picture is as dark as some writers have drawn it.

If one can trust the literature and his own experiences, today's college is a far cry from the lofty, academic citadel it was a generation or two ago. Then, only the cream-of-the-crop were in attendance. Since that time, not only has the *number* but also the *per cent* of high school graduates going on to college increased. Thus, today, the college population may be divided roughly into three groups:

1. Those who were not particularly strong students in high school, who may or may

not have definite academically-trained goals in mind, and who are in college largely because "it is the thing to do." These "students" have always been with us but their number has increased enormously and, with a democratic philosophy of education, will probably continue to grow.

2. The ordinary serious-minded, scholarly, college student.

3. The gifted student who is getting more and more attention and who may eventually come to be considered as important as the star athlete in building the reputation of his alma mater.

To say that there are many ill-prepared individuals now in college is not only to recognize the "new trend"—a trend faced by the high schools years ago—but may represent cognizance of pupil weaknesses which have always existed but were submerged in a policy of admission by transcript. Most of the ill-prepared do not elect courses in college mathematics but are casualties of requirements of one type or another.

What of the ordinary, average, studious individual who has graced our campuses for the past fifty years? Since you and I were members of that group, we may be inclined to glamorize the success we enjoyed and to discount, from this distance, such weaknesses as were inscribed on our records. Yet, does anyone seriously believe that today's students of college algebra, trigonometry, and analytic geometry are the equals of those of a generation ago?

Two years ago an opportunity arose at the State University of Iowa to gather a bit of evidence relating to this question. Approximately forty objective examinations, containing more than seven hundred items, had been prepared and administered during the 1937–1938 academic year to all of the students enrolled for the regular algebra, trigonometry, analytic geometry sequence. The results had been studied systematically and indices were available for all tests and items. In spite of the fact that the courses of study had changed somewhat, sufficient acceptable items were available to prepare tests for the 1953–1954 classes. In this manner it was possible to make direct comparisons between the 1938 and 1954 groups, both testwise and itemwise.

As might be expected, each group excelled in the items which most nearly characterized the emphases prevalent at the time the tests were taken. In general, however, no over-all advantages could be ascribed to either.

The results, of course, prove nothing. Indeed, the study was undertaken only to satisfy a casual interest and was not designed to produce scientific measures of comparison. There is an *indication*, however, that the students of a generation ago, whose abilities undoubtedly have been magnified by the span of sixteen years, may have been no better students of mathematics than their modern counterparts. Perhaps the high school of today is preparing these typical students for college, not as well as we would like, but as well as at any time in the past.

And what of the gifted pupil—the one who is all-important to the welfare of the nation in terms of national survival? We are unquestionably failing to find and stimulate many of these potential leaders. On the other hand, the evidence I have seen indicates that the topflight students who are graduated from today's high school are much better prepared than those of any other generation in history. Why shouldn't they be? You taught them, and you are numbered among the best informed teachers in the profession—the members of the National Council and readers of THE MATHEMATICS TEACHER.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Arithmetic: Its Structure and Concepts, Francis J. Mueller, Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1956. Cloth, xv+279 pp., $5.50.

Designed primarily for elementary teachers (prospective and in-service) this book aims to provide an exposition of the structure and concepts of arithmetic in mature terms. A unique feature of the book is its organization based upon three fundamental activities and their attending processes. These activities are synthesis, analysis, and comparison. The processes of addition and multiplication are subsumed under synthesis, and subtraction and division under analysis. Furthermore, the activity of comparison is carried out with subtraction and division. Quite a bit of historical development is included where pertinent, as well as discussions of systems of notation other than base ten. Sections giving algebraic interpretations of notation and the four processes are also provided.

The author is to be commended for writing a book which attempts to clarify the concepts and skills of arithmetic. This kind of book is urgently needed today. The sections dealing with historical perspective are probably the best ones in the book. Also, many instructors will welcome the exercises appearing at the end of each unit.

On the other hand, this book has many weak features in the opinion of the reviewer. The most serious of these is that the exposition is highly verbal and needlessly abstract. The reviewer readily grants that the concepts, principles, and skills of arithmetic should be explained and discussed in mature terms, but this does not preclude extensive use of diagrams and sketches of situations involving real objects. In other words, to help the mature person understand arithmetic, the author uses explanations and situations which are not easily adaptable to classroom use. For this reason many students have difficulty in understanding the book.

The reviewer also questions the way in which the four fundamental operations are defined. For example, addition is defined, on page 49, as "the process of finding, without counting, the number property of a group which is formed by the combination of two or more groups." This in effect identifies addition with the addition algorithm used in computation, whereas addition ought to be identified as the activity of combining groups of known sizes. Similar defects are present in the definitions of the other basic operations.

One other weakness ought to be mentioned—the author fails to develop the meaning of a number explicitly. As a result, a distinction is made between "concrete numbers," "denominate numbers," and "abstract numbers"—a confusing and meaningless distinction.

In summary, the reviewer feels that this book is a step in the right direction but it is only partially successful.—*Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa.*

Differential Calculus, William Ferrar, New York: Oxford University Press, 1956. Cloth, 296 pp., $4.40.

This text is suitable for a college senior who wishes to know something about real variable theory or for a first year graduate student who wishes to brush up his basic understandings before going into a graduate course in function theory. The author presupposes that the reader has had a course in differential and integral calculus and is now ready to take a second and a more critical look at the subject matter of the differential calculus.

Reading this book would serve a twofold purpose. Not only would the student re-examine the fundamental notions of limit, continuity, derivative, partial derivative, etc., but he would in the course of his investigations learn to read and to work with the symbolism of modern mathematics. The language of mathematics is developed by easy stages and not presented, as is too often the case, in one large indigestible lump.

Wherever possible, basic theorems are proved. The Heine Borel theorem is developed and used to study the properties of continuous functions. If the proof of a particular theorem is rather complicated and not especially pertinent, a reference is cited where the proof may be found. A good many references are made throughout the text, and the serious student has a splendid opportunity to gain at least a nodding acquaintance with many of the standard works in the literature of function theory.

The references also enable him to track down easily more information on any topic which may intrigue him.

The presentation is formal throughout and quite different from the breezy treatment of a good many undergraduate texts. The student who uses this book can not only learn more mathematics but he can also practice reading modern mathematical language at a not too difficult level.—*Augusta Schurrer, Iowa State Teachers College, Cedar Falls, Iowa.*

Practical Trigonometry, Ralph S. Underwood and Horace E. Woodward, Boston: Houghton Mifflin Company, 1956. Cloth, ix+251 pp., $3.25.

As the word *Practical* in the title seems to suggest, the text places a major emphasis upon some of the applications of trigonometry.

There is considerable stress on rounding off the results of calculations with numbers that arise from measurement. This is to be commended, however bad advice is occasionally given. For example, it is suggested that in multiplication the factors should first be rounded off to the number of significant digits of the least accurate factor.

The definition of a variable as a quantity which may be varied in size seems inadequate. The word *function* is used consistently throughout the text for what most writers would call the dependent variable of a function.

The graphs of the trigonometric functions appear near the end of the book. Thus through most of his work the student is deprived of a valuable visual representation of the relationships involved. The periodic nature of the functions is mentioned at this point, but there is no hint of the extensive applications of these mathematical models to repeating type phenomena.

In general the text is well written. The problems, particularly those dealing with applications, are better than usual. The figures are well drawn and clearly labeled. Four-place tables of the trigonometric functions, five-place tables of logarithms, and five-place tables of logarithms of the trigonometric functions are included.

The instructor who wishes to emphasize numerical trigonometry and solution of triangles will find this book a good text. For a course with an emphasis on analytic trigonometry with some applications to periodic phenomena, the book is less satisfactory.—*Fred W. Lott, Iowa State Teachers College, Cedar Falls, Iowa.*

FILMSTRIP

Using and Understanding Numbers, Decimals, and Measurements, "Meaning and Reading of Decimals," "Addition and Subtraction of Decimals," "Multiplication of Decimals," "Division of Decimals," "Changing Fractions to Decimals—Decimals to Fractions," "Advancing in Linear Measurement," "Advancing in Quantity Measurements." The Society for Visual Education, Inc., 1345 Diversey Parkway, Chicago 14, Illinois. 35 mm., color filmstrips prepared by Joseph J. Urbancek and Francesca C. Urbancek. $5.50 each or $35.00 per set.

This series of filmstrips is done in attractive color and is illustrated by situations which upper elementary and junior high school pupils should find interesting. Each strip begins by telling the teacher what concepts are presented in the strip. This is followed by sensible instructions about the utilization of the strip. Most of the frames present a situation that involves decimals or measurements. These situations involve everyday incidents such as traveling, shopping, comparing temperatures, producing food, making clothing, and playing games. Most frames pose questions or make suggestions for pupil discussion. Many frames present the meaning of place value or units of measure. Other frames explain computational procedures much like a modern arithmetic textbook. Each strip ends with a summary and problems for practice or questions to be answered by the viewer. The content of the strips is indicated by the titles given above. Each strip presents ideas, computational processes, and problems involved in each topic.

It is usually easy to criticize classroom materials which someone else has found successful. In this series, for example, it seems that most frames are verbally similar to material presented in modern textbooks or by the teacher on the blackboard. However the use of color and the dramatic presentation by projection should lend needed variety to the typical arithmetic lesson. The color is attractive but is seldom used to full advantage to focus attention on common elements. For example, place value might be emphasized by using the same color for a given place in a series of examples. Most computational processes are presented in the typical manner, for example, borrowing and the use of carats in long division, rather than approximating answers or leading the pupil to discover the process.

One of the greatest assets of this series is the involvement of the viewer in solving problems and answering questions. The photographic quality of all seven strips is excellent, as is the appropriateness of the illustrations. Pupils are likely to enjoy the measurement of a dog as an illustration of the yard unit of linear measure or the application to games and contests. This series of filmstrips will probably find greatest value in the sixth and seventh grades.—Donovan A. Johnson, *University of Minnesota, Minneapolis, Minnesota.*

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia

Giving the student a part in his evaluation

by Joseph N. Payne, University of Wisconsin, Madison, Wisconsin

David Smith, a junior at Central High School, made C in algebra on his last report card; Jane Arnold made B, and Tom Jones made A. Seeing these grades, what did David, Jane, and Tom conclude about their progress in learning algebra? David said that he needs to do better on tests; Jane concluded that her homework is not too good; Tom decided that he is excelling in all phases of algebra. How accurately and how thoroughly have these students appraised their progress in mathematics? How closely did David and Jane locate their weaknesses in algebra? Was Tom right when he decided that his A meant that he is excelling in all aspects of the subject?

The appraisals of David, Jane, and Tom may help each to maintain or to improve his algebra grade. But this is more likely to happen if the achievement these grades represent is carefully analyzed by the pupils themselves. As a supplement to their grades on a report card, I believe students should make for themselves a systematic analysis of their progress in mathematics. Such an analysis will be likely to improve their grades, and it will also broaden their understanding of mathematics. This result is likely because a systematic self-analysis of a pupil's work will help to reveal specific topics which should receive more attention or work habits he needs to improve.

I have used a check list of items such as the one below, to help high school students *think* about their progress in mathematics. The check list contains three major divisions: (1) Study of Organized Subject Content; (2) Study Skills and Work Habits; and (3) Class Participation.

Under each division, there appear several statements to be checked by the student. As I use the check list, it is duplicated and each student given a copy. At intervals of from four to six weeks, each student is asked to think reflectively and objectively about his work. As he thinks about his work, he should examine tests, homework papers, and the like for evidence on the various items in the check list. After giving careful thought to his work, he checks the items. (By arranging the check list with five or six blanks to the right of each item, the same form can be used for the entire year, thus showing progress from one evaluating period to the next.)

After the student checks the items, the teacher goes over them carefully and notes the ones which he thinks the student has incorrectly evaluated. Differences between the student's and the teacher's evaluation become the basis of a conference between pupil and teacher. If time permits, the teacher should arrange a short conference with each student to talk about the evaluation. If this is not possible, the teacher should confer with the pupils needing most attention.

STUDENT CHECK LIST

MY PROGRESS IN MATHEMATICS

After a careful analysis of my work in mathematics since the last evaluating period, I rate myself as follows:

A. CONTENT I AM LEARNING

1. I can use mathematical concepts and skills in varied problems (a) almost always (b) sometimes (c) seldom.
2. My thinking in mathematics is (a) good (b) average (c) poor.
3. The quality of my daily work is (a) good (b) fair (c) minimum.
4. The quality of work on my tests is (a) good (b) fair (c) minimum.
5. I make errors in computation (a) frequently (b) sometimes (c) seldom.
6. I have trouble solving verbal problems (a) frequently (b) sometimes (c) seldom.

B. STUDY SKILLS AND WORK HABITS

1. I take pride in doing good work (a) always (b) sometimes (c) seldom.
2. My written work is neat (a) always (b) sometimes (c) seldom.
3. I make good use of study time in class (a) always (b) sometimes (c) seldom.
4. I try to understand what I do instead of merely memorizing (a) always (b) sometimes (c) seldom.
5. I act before thinking through a process (a) always (b) sometimes (c) seldom.
6. I have trouble reading and understanding explanatory materials (a) frequently (b) sometimes (c) seldom.

C. CLASS PARTICIPATION

1. I disturb study situations (a) often (b) seldom (c) never.
2. I am attentive during class discussions (a) always (b) sometimes (c) seldom.
3. I make pertinent contributions to class discussions (a) often (b) sometimes (c) seldom.
4. I bring my personal equipment to class as needed (a) always (b) frequently (c) seldom.
5. I take initiative in bringing new ideas and materials to class (a) frequently (b) sometimes (c) never.

These are only some of the items I use with my students. Other teachers will doubtless have other items they prefer to use. After your check list has been developed, it is not necessary for students to check all the items at each evaluating period. When there has been no evidence accumulated on a particular item, it should not be checked.

A simple experiment in algebra

by H. C. Trimble, Iowa State Teachers College, Cedar Falls, Iowa

Do you ever wish for a simple experiment that suggests a generalization for your algebra students to prove? Here is one you may like.

Take a cloth measuring tape 36 inches long. Use pins on a drawing board to mark the corners of a rectangle whose perimeter is 36 inches. As you change the length (L), from L more than 0 to L less than 18, and hence the width ($W = 18 - L$), you get different areas as follows:

L (in.)	W (in.)	A (in.^{2})	$81 - A$ (in.)
3	15	45	36
6	12	72	9
9	9	81	0
11	7	77	4
15	3	45	36
17	1	17	64

Notice that the square has the greatest area. Notice that the difference $81-A$ between the area of the square and the area of a rectangle always seems to be a perfect square. Students can make these measurements themselves. They can check values of $81-A$ that correspond to integral values of L. Is the value of $81-A$ always a perfect square? For example, when $L=15.4$, $W=2.6$, $A=40.04$, $81-A=40.96=6.4^2$. But, is this a general result? Yes, because it can be seen that $81-A=81-LW=81-L(18-L)=81-18L+L^2=(9-L)^2$.

SIAM—the newest mathematical society

Continued from page 64

and that rapid expansion of industrial research had created a need for basic analytical thought and new mathematical methods. Many mathematicians, engineers, physicists, and others, felt that application of mathematics to problems in business must be promoted in a positive fashion in order that new technological questions can be more adequately considered. It therefore seemed essential to form an organization such as ours to stimulate the growth of the use of mathematics.

SIAM attempts to achieve its aims through its publications and its technical meetings, as well as by other means. The Society publishes a quarterly journal as well as a newsletter which has ten issues per year. The Journal publishes research papers which contain or lead to new mathematical techniques, expository papers on mathematical techniques and their applications, survey papers, and papers calling attention to areas in which there is need for mathematical development. The Newsletter publishes book reviews, notices of various kinds of applied mathematical activities, and other information of general interest. For example, in the February 1956 issue, the Newsletter included the full text of a speech entitled "Science and Engineering in the USSR and the USA: A Comparison and Appraisal," by Dr. M. H. Trytten, Director of the Office of Scientific Personnel, National Research Council. In addition, the Society is publishing in April as its Monograph No. 1, "Survey of Training and Research in Applied Mathematics in the United States," prepared by F. J. Weyl for the National Research Council under contract with the National Science Foundation.

SIAM has three classes of membership: Individual, Academic, and Corporate. Academic membership is open to any nonprofit organization, whereas Corporate membership is available for profit-making organizations. SIAM's current roster of individual members includes over 1,200 names of persons in the United States and foreign countries. Groups of members living in the same geographical area have the privilege of petitioning for formation of sections of SIAM. At present, the Society has operating sections in the following areas: Baltimore, Boston-Cambridge, Central Pennsylvania, Delaware Valley, New York, Northern California, and Southern California. In addition to sponsoring technical meetings, the sections also sponsor informal activities such as study groups. All activities of the Society are open to the public.

SIAM is anxious to co-operate with organizations having similar aims, and particularly welcomes opportunities to participate in jointly-sponsored meetings. In the past, it has co-sponsored or co-operated in meetings on national or local levels with the American Mathematical Society, the Association for Symbolic Logic, the Mathematical Association of America, the Institute of Radio Engineers, the American Institute of Electrical Engineers, the American Society for Engineering Education, the Association for Computing Machinery, The Institute of Management Sciences, and the Operations Research Society of America. Among other activities of our organization are joint sponsorship of the periodical *Mathematical Reviews*, membership in the Mathematics Division of the National Research Council, associate membership in the American Association for the Advancement of Science, and joint sponsorship of the Mathematical Sciences Employment Register which is available at national meetings of the various mathematical societies.

Our organization is also interested in the promotion of scholarships in mathematics for high school graduates. Indeed, at its meeting on October 26, 1955, the Council of the national organization specifically stated that the improvement of secondary school instruction in mathematics and science and the stimulation of interest among capable students toward scientific study through an industry-supported scholarship program are consistent with the purposes of SIAM. In particular, the Council voted to support, through publicity and a modest financial contribution, programs of this nature in which NCTM is interested. The Society believes that it is in a very favorable position to assist such causes because of its industrial contacts.

Additional information about SIAM may be obtained by addressing the Society at Box 7541, Philadelphia 1, Pennsylvania.—Thomas R. Southard, University of California, Los Angeles, California.

• TESTING TIME

Edited by Robert S. Fouch and Robert Kalin, Florida State University, Tallahassee, Florida

A new department

With this issue, THE MATHEMATICS TEACHER inaugurates a new department devoted to discussions of evaluation in secondary mathematics. It is to be a practical department, not one consisting solely of abstract arguments on the Theory of Testing—but with the reservation that occasionally there's nothing so practical as a good theory.

However, we have promised to make the section useful to mathematics teachers. In order to do this successfully we need several kinds of reactions from you, the readers.

1. Reactions on what you need and what you want to find in this column. E.g., do you want reviews of currently available standardized tests; do you want discussions on objective versus essay type tests; do you want references to articles elsewhere on testing?

2. Your evaluations of what appears here now and in the future.

3. Your contributions for publication here; for, like other departments, this one will be entirely *for* you and also largely *by* you.

These contributions may be small (perhaps a single test question that you have found especially good) or large (perhaps a whole test complete with specifications of objectives to be tested and analysis of results as it was actually given). The important thing is that we must have contributions if we are to have a department.

In order to get started, we decided that we would take a standard unit in high school algebra and consider how a teacher might construct a test for use on completion of that unit. Here is such a test.

1. log 938 = _____
2. log .0000561 = _____
3. log _____ = 3.4382
4. log _____ = −2+.5699
5. To multiply two numbers by logarithms, we _____ the logarithms of the two numbers.
6. To divide two numbers, we _____ the logarithms of the one number from the logarithm of the other.

In problems 7–10, compute by means of logarithms.

7. $8{,}650 \times .934$
8. $761 \div 5{,}620$
9. $(23.8)^3$
10. $\sqrt[4]{89.3}$

This is a paraphrase of the first effort in test construction of one of our own teacher trainees who had been teaching a unit on logarithms. Since he had not yet taken a course in test construction, we suspect that this test reflects his own experience in being tested as a student. What is your opinion of the merit of this test? Good, bad, or indifferent?

Like any other judgment, this judgment must be made on the basis of certain principles, assumptions, and criteria. One must ask certain more detailed questions. We suggest these two questions:

1. What mathematical principles, understandings, abilities, or skills are actually tested?

2. What student behavior is actually measured?

We also suggest two basic principles about tests:

1. A test is good (or bad) to the extent that it measures (or fails to measure) the attainment of the objectives of instruction set down by the teacher.
2. As a corollary to this, since we believe that understanding is important, a test should measure student understanding of mathematical principles and student ability to apply these understandings, as well as simpler skills.

Now back to our sample test. Does it meet our first basic principle? It fails in several instances because of routine technical matters. For example, Item 6 has internal evidence which makes it possible for a student to answer correctly without even knowing what a logarithm is. Second, one might quibble about omitting the base (certainly 10) in the logarithm notation; the "wise" student might claim the given questions cannot be answered because the base is missing. Third, the student has not been directed to show his work in Items 7–10. Since each of these items involves many skills, the teacher cannot analyze the failure of the student who merely writes down the wrong answer.

But a more important issue than such technical matters is an analysis of the objectives actually tested. Briefly, this test appears only to measure the following skills: (1) using tables; (2) finding characteristics; (3) quoting the product and quotient rules; (4) following the procedures for the four fundamental operations.

Under the hypothetical circumstance that the course objectives were nothing more than teaching such skills, this is a satisfactory test according to our first basic principle. Though satisfactory, we still don't like this test. Our second principle forces us to consider other objectives in teaching and testing this subject matter. In brief, we feel that it would be possible to get a perfect score on the basis of rote learnings and mechanical manipulations, and that very little understanding would be required.

What do we mean by understanding, and how do we test for it? We would like to suggest that understanding is revealed when the student is able to *apply* a concept or a principle in a *novel* situation. Novelty is, of course, a function of the learning experiences the student has had. For example, consider Item 10. If the class had *not* studied extraction of roots by logarithms, then this item would be novel and it could test understanding by requiring the student to apply his knowledge of logarithms combined with his knowledge of fractional exponents.

To carry this idea further, suppose we wished to test a real understanding of characteristic as opposed to mere skill in counting the number of digits. Consider this question:

What is the characteristic of $\log_2 27$?

The student who has merely memorized the rule for characteristic for base 10 might very well give "1" as his answer rather than "4."

Or suppose one wished to test the understanding that logarithms are exponents. We propose such items as these:

What is $\log_3 1$?

What is $\log_{14} 14$?

What is $\log_5 25$?

If $\log_a 5 = 3$ and $a^x = 5$, then $x = ?$

Here it seems that the student must apply an understanding that $\log_b n = p$ means that $b^p = n$, if he is to arrive at the correct answers of 0, 1, 2, and 3, respectively.

In similar vein, one could get at the product, quotient, power, and root rules by giving students the value of $\log_6 3$ and $\log_6 4$, and then asking him to give the values of $\log_6 12$, $\log_6 4/3$, $\log_6 9$, $\log_6 2$.

Two comments seem necessary at this point. First, if the student has done specific exercises involving base 2, 3, 5,

14, etc.,—items somewhat like these above—then understanding would not be measured by the above items. For, to repeat an earlier thesis, understanding is revealed only if the student is required to apply a concept in a novel situation. In suggesting such questions, we assume therefore that the student is fully aware of the relationship between the logarithmic and exponential functions, has certainly constructed partial tables of logarithms to the base 10 and perhaps one other base as well, but has not met head-on with our suggested question situations in extensive periods of practice.

The second comment is that the questions suggested thus far deal with quite specific understandings. (These questions might very easily be framed in multiple-choice format.) It seems essential that any test contain at least one item that requires the student to display a broader understanding of the subject matter.

For such purpose, we suggest this item in three parts:

a. Make a rough sketch, on the same co-ordinate axes, of $y = \log_{10} x$ and $y = \log_{0.1} x$, for values of x between 0 and 10. Write a brief paragraph discussing the relationship of these two functions.

b. When interpolating with logarithms to base 10, is the interpolated value less than, more than, or the same as the actual value? Why?

c. When interpolating with logarithms to base 0.1, is the interpolated value less than, more than, or the same as the actual value? Why?

To answer such a question seems to require a rather broad concept of logarithms. It is one that could probably appear in essay form only. In proposing it, our purpose is illustrative only, and we apologize for leaving unsettled the precise wording and scoring of problems involved. But space is limited; perhaps you, the reader, would care to contribute an article devoted to a discussion of such problems.

Before concluding this first edition of "Testing Time," we should like to propose a situation which might lead to another novel question to test student understanding.

By analogy to the usual definition of logarithms ("$\log_b n = p$" means "$b^p = n$"), we introduce the notion of kogarithms which are defined by the relation:

"$\text{kog}^p n = b$" means "$b^p = n$".

(We may read "$\text{kog}^p n$" as "kog, upper p, of n".) For example, what is $\text{kog}^3 8$? Answer: 2. In general, what are the properties of kogs as compared with those of logs?

We hope that some of you will mull over this proposal and perhaps send us a contribution based thereon.

In conclusion, how would or do *you*, the reader, measure understanding? What is your reaction to the main theses of this first article? And finally, "Contributions, anyone?"

I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations" are simply our notes of our observations.—*Hardy*, A Mathematician's Apology.

Report of the Nominating Committee

The Committee on Nominations and Elections presents below the names of persons nominated for offices to be filled in the election of 1957. The term of office for the Vice-Presidents is two years. Three directors are to be elected for terms of three years.

Attention is called to the following provisions adopted by the board of directors.

"Nominations shall be made so that there shall be not more than one director elected from each state, and that there shall be one director, and not more than two, elected from each area." For a map of the regions as now constituted, members may consult THE MATHEMATICS TEACHER for October, 1955, page 442. Under the regulation quoted above it is necessary to insure the election of at least one director each from the Western and Southeastern Regions. To avoid a situation in which nomination would be equivalent to election, the Committee has nominated two candidates from the Western Region and two candidates from the Southeastern Region, and has adopted the following rule for determining who shall be declared elected as directors:

The three candidates receiving the largest number of votes will be declared elected, provided the Western and Southeastern Regions are each represented among these three candidates. If either of these two regions should not be so represented, then the candidate, within that region, receiving the larger number of votes will be declared among those elected.

Ballots will be mailed on or before February 6, 1957 from the Washington office to members of record as of that date. Ballots returned and postmarked not later than March 6, 1957 will be counted.

The Committee wishes to thank all those who suggested persons for consideration and urges all members to vote.

F. LYNWOOD WREN, Chairman
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MAURICE HARTUNG
JANET HEIGHT
CLARK LAY
ZEKE LOFLIN
JOHN MAYOR
BESS PATTON
PHILIP PEAK

NOMINEES FOR VICE-PRESIDENT—COLLEGE LEVEL

ROBERT E. PINGRY

HENRY W. SYER

Robert E. Pingry

Associate Professor of Mathematics and of Education, University of Illinois.

A.B. and A.M., Ball State Teachers College; Ph.D., Cornell University.

Teacher of Mathematics, Mitchell, Indiana High School; Teacher of Physics, Chemistry, and Mathematics, Griffith, Indiana High School and Junior High School; Instructor in Mathematics and Navigation, U.S. Naval Reserve Midshipman's School, Cornell University; Navigator of an aircraft carrier on Pacific duty; Instructor in Mathematics, Bowling Green State University; Teaching Assistant in Mathematics, Cornell University; Visiting Lecturer in Education and in Mathematics, University of Colorado.

Member: NCTM, Mathematical Association of America, Illinois Council of Teachers of Mathematics, Phi Delta Kappa, Phi Kappa Phi, Kappa Delta Pi, AAUP.

Activities: Member Joint Committee of NCTM and MAA on teacher education in mathematics, 1953–54. Member of NCTM committee on research in algebra, 1953–55. Editor of *Newsletter* of Illinois Council of Teachers of Mathematics. Member of Executive Board of Illinois Council of Teachers of Mathematics. Member of Committee on Contests and Awards of the Illinois Section of the Mathematical Association of America. Numerous speeches and papers at mathematics conferences, teachers' institutes, and National Council meetings.

Publications: Coauthor, "Problem Solving in Mathematics," Chapter 8, *Twenty-First Yearbook* of NCTM. Coauthor of a series of junior high school textbooks in general mathematics, *Using Mathematics*, 7–9. Articles in THE MATHEMATICS TEACHER, *Journal of Experimental Education*, and *The Education Digest*.

Henry W. Syer

Associate Professor of Education, Boston University, Boston, Massachusetts. In charge of Teaching of Mathematics, Audio-Visual Education, and Director of Travel Courses.

B.S., M.A. and Ed. D., Harvard University.

Instructor in mathematics and science, Gunnery School, Washington, Connecticut; Instructor in mathematics, Culver Military Academy, Culver, Indiana; U. S. Army, Anti-aircraft Artillery (Computer Section) and Signal Corps (Photographic Section); Assistant and associate professor of education, School of Education, Boston University; Visiting professor, School of Education University of Michigan.

Member: NCTM, Association of Teachers of Mathematics in New England, NEA, AAAS, MAA, Phi Beta Kappa, Phi Delta Kappa. Listed in: *Leaders in Education*, *Who's Who in Education*, and *Who's Who in the East*.

Activities in NCTM: Coeditor, "Aids to Teaching" Department, THE MATHEMATICS TEACHER, 1948–1954; Committee on Evaluation of Films and Filmstrips, 1948–1957 (Chairman, 1948–1951); Committee on Institutes, Workshops, and Conferences, 1948–1957; Board of Directors, 1949–1952; Associate editor, THE MATHEMATICS TEACHER, 1950–1952; Chairman, Committee on Supplementary Publications, 1950–1955 (Called "Committee on Publications of Current Interest," 1950–1953); Committee on Evaluation and Research, 1951–1953; Program Committee, 12th Summer Meeting, held jointly with N.E. Institute, 1952; Committee on Materials, Summer Meeting, 1954; Yearbook Planning Committee for "Basic Concepts in Mathematics," 1954–1957; Committee on International Cooperation in Mathematics, 1954–1957 (Chairman, 1956–1957); Co-operative Committee on Science and Mathematics of the AAAS, 1954–1957; Planning Committee for Local Arrangements, Annual convention, Boston, 1955; Participant in conventions (speaker, leader of film forums, demonstration lesson, science demonstration, etc.), 1938–1957.

Other Activities: in Association of Teachers of Mathematics in New England: Organizer and first general chairman, 1949; Member of Council, 1950–1955; Chairman, Committee on Publicity and Publications, 1950; Committee on Mathematical Games and Puzzles, 1950–1951 (Chairman, 1951); Program Committee, 1952–1955; Mathematics Contest Committee, 1952–1953; Vice-President, 1954–1957; Chairman, Committee on Membership and Hospitality, 1956–1957. Other activities: Consultant to Commission on Motion Pictures, American Council on Education, 1947–1948; Committee on Mathematics for Journal of American Association of Colleges of Teacher Education, 1953; United States sub-committee of International Instruction Committee for Mathematics of International Union of Mathematics, 1952–1955; Committee on National Mass Media Awards for Thomas Alva Edison Foundation, 1955; Committee on Recruitment and Guidance Material for Science and Mathematics, AAAS, 1955–1957; Representative, Council on Student Travel, 1955–1957; Examiner in Audio-Visual Education, Massachusetts Civil Service, 1951–1955; Tour leader, Boston University Travel Courses, 1953 (Mathematics Tour), 1955; European Coordinator, Boston University Travel Courses, 1956; Curriculum consultant in mathematics; Consultant on filmstrips.

Publications: Numerous articles in: *School Science and Mathematics*, THE MATHEMATICS TEACHER, *The Philosophical Forum*, *Film News*, *Audio-Visual Guide*, and chapters in *Yearbooks* of NCTM.

NOMINEES FOR VICE-PRESIDENT—JUNIOR HIGH SCHOOL LEVEL

ALICE M. HACH

ELIZABETH JEAN ROUDEBUSH

Alice M. Hach

Consultant in Mathematics, Racine Public Schools.

B.A., University of Iowa; M.A., University of Michigan.

Teacher, Junior High School Mathematics, Stuart, Iowa, Fort Dodge, Iowa, Racine, Wisconsin, Ann Arbor, Michigan; Critic teacher, University High School, Ann Arbor, Michigan; Consultant in Mathematics, Racine Public Schools.

Member: NCTM; CASMT; NEA; Wisconsin Education Association; Michigan Council of Teachers of Mathematics; Women's Mathematics Club of Chicago and Vicinity; Association of Childhood Education; Wisconsin Mathematics Council; Pi Lambda Theta; Delta Kappa Gamma.

Activities in NCTM: Member of Place of Meetings Committee; Editorial Board of forthcoming Yearbook; Cochairman for local arrangements of summer meeting of NCTM, Kalamazoo, Michigan, 1953; Committee member of the mathematics number of the *National Association of Secondary School Principals Bulletin*, May, 1954; Cochairman for local arrangements of annual meeting of NCTM, Milwaukee, 1955; Appearance on convention programs.

Other activities: President and Vice-President of Iowa Mathematics Association; Member of Curriculum Committee for State of Iowa; Chairman of Mathematics Section for Michigan Schoolmasters' Club; Member of Board of Directors for Michigan Council of Teachers of Mathematics; Chairman of Mathematics Section for Michigan State Association; Director of Arithmetic Summer Workshops, Eastern Oregon College of Education, LaGrande, 1952, 1953; Oregon College of Education, Monmouth, 1954; Staff Member of Summer Institutes, Mathematics Teachers of New Jersey, Rutgers University, 1955, New England Teachers of Mathematics, Williams College, 1956; President of Pi Lambda Theta, Beta Chapter; Member of Resolutions Committee for CASMT, Certificate Award from Quiz Kids for being one of most popular classroom teachers in nation; Summer Camp Conservation Scholarship, Ann Arbor Michigan Garden Club; Treasurer of Delta Kappa Gamma —Beta Chapter; Listed in *Who's Who in American Education.*

Publications: *Twenty-Second Yearbook of NCTM*: "Reporting to Parents Pupil Progress," "Importance of Early Guidance at Junior High School Level"; THE MATHEMATICS TEACHER: "A Project Giving Practice in Reading and Writing Roman Numerals," "As Others Saw Him"—The Schorling Number; *Junior Scholastic*: "Junior Scholastic Brings a Fresh Approach to Problem Solving."

Elizabeth Jean Roudebush

Director of Mathematics from Kindergarten through Grade Twelve for Seattle Public Schools, Seattle, Washington.

Graduate of State College of Washington, Pullman, Washington: A. M., Teachers College, Columbia University; summer sessions at University of Washington, University of California at Los Angeles, and University of Southern California.

Teacher, Mathematics, Roosevelt High School, Seattle, Washington; Mathematics Department Head, Edison Technical School, Seattle, Washington; Director of Mathematics, Seattle, 1949—. During World War II served in WAVES as Women's Reserve Representative at U. S. Naval Hospital in Chelsea, Massachusetts.

Member: NCTM; ASCD; NEA; Washington Education Association (Past President of local group affiliated with WEA and NEA); Delta Kappa Gamma (Past Vice-President of local chapter); Pi Lambda Theta (Past President of local alumnae chapter).

Activities in NCTM: Member, Board of Directors, 1953–56; Regional Representative of Western Region of Affiliated Groups; Chairman of Place of Meetings Committee; Cochairman in Charge of Local Arrangements for 1954 Summer Meeting; Section Speaker at 1953 Christmas Meeting; Chairman of Affiliated Groups 1956–59.

Other Activities: helped organize Puget Sound Council of Teachers of Mathematics; helped organize Washington State Mathematics Council.

Publications: *Laboratory Geometry*; "An Arithmetic Bulletin for Parents," THE MATHEMATICS TEACHER, May, 1951.

NOMINEES FOR THE BOARD OF DIRECTORS

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ROBERT E. K. ROURKE

HELEN L. GARSTENS

ANNIE JOHN WILLIAMS

LOTTCHEN LIPP HUNTER

Clifford Bell

Western Region

Professor of Mathematics and Head of Mathematics Extension, Los Angeles Campus, University of California.

B.S. and Ph.D., University of California, Berkeley.

Teaching Fellow, University of California Berkeley; Instructor to Professor, University of California, Los Angeles; Educational Supervisor in Charge of Mathematics Courses, Southern California area, University of California ESMWT program. In charge of teacher-training courses in Department of Mathematics, University of California, Los Angeles.

Member: NCTM, California Mathematics Council, NEA, MAA, AMS, AAAS, and a number of honor societies. Listed in *American Men of Science, Who's Who in*

Education and other biographical books.

Activities in NCTM: Membership Committee; Institute Committee; Joint Committee on Co-operation with Industry and the later Committee on Co-operation with Industry; Curriculum Committee; Publications Board; Nomination and Election Committee (1955); Chairman, Local Arrangements Committee, Fourteenth Christmas Meeting in Los Angeles; Co-chairman, Sixteenth Summer Meeting in Los Angeles.

Other activities: Chairman of the Planning Committee of the Annual California Conference for Teachers of Mathematics and Director of the Conference.

Coauthor of textbooks in trigonometry, commercial algebra, and the mathematics of finance. Author of articles in the *Bulletin* of the American Mathematical Society, *The American Mathematical Monthly*, THE MATHEMATICS TEACHER, *The Arithmetic Teacher*, other national and foreign journals.

Sylvia Vopni

Western Region

Assistant Professor of Education, University of Washington, Seattle, Washington.

B.A., M.A., Ph.D., University of Washington.

Teacher of mathematics and science, public schools of Washington State; physicist, Navy Department; teacher of mathematics and physics, Edison Technical School; Northwest Regional Soroptimist Fellowship in Education; head of department of mathematics and science, Edison Technical School; Assistant Researcher in Education, University of Washington, also teaching summer school and extension division classes in education and mathematics; Acting Assistant Professor of Education.

Member: NCTM; MAA; AAAS; American Statistical Association; Washington Council of Mathematics Teachers; NEA; ASCD; Washington Education Association; American Association of University Women; Phi Beta Kappa; Pi Lambda Theta; Delta Kappa Gamma; Pi Mu Epsilon.

Activities in NCTM: Cochairman on Local Arrangements, Fourteenth Summer Meeting, 1954; discussion leader or participant on NCTM programs, 1949, 1953 1954, 1956; member of committees on television, scholarships, and summer conferences. Western Vice-President, Washington Council of Mathematics Teachers, 1955– .

Other Activities: President, Seattle Association of Classroom Teachers; president, Omicron Chapter, Delta Kappa Gamma; national vice-president, Pi Lambda Theta. Participant in several mathematics conferences and summer institutes.

Publications: Associate editor, *Pi Lambda Theta Journal*, 1946–49; feature editor, *Educational Horizons*, 1954–55; "Nation's Mathematics Teachers Coming West." *Washington Education*, 1954; "Resources in the Teaching of Elementary School Science." *College of Education Record*, March, 1955; "Mathematics and the Humanities," *College of Education Record*, May, 1955.

Robert E. K. Rourke

Northeastern Division

Head of Department of Mathematics, Kent School, Kent, Connecticut.

Honour B.A., Queen's University, Kingston, Ontario; A.M., Harvard; course work for doctorate in mathematics, Harvard.

Teacher, Head of Mathematics Department, Headmaster, Pickering College, Newmarket, Ontario.

Activities: Past President of Ontario Association of Teachers of Mathematics and Physics; former executive of Canadian Mathematical Congress; member of Commission on Mathematics of CEEB.

Publications: Coauthor of *An Advanced Course in Algebra; Trigonometry and Statistics; Mathematics for Canadians*, Books I, II, III; articles in professional periodicals.

Helen L. Garstens

Southeastern Region

Assistant in Charge of Mathematics, in Office of Secondary Supervision, in Arlington County, Virginia.

Graduate of Hunter College; postgraduate work in mathematics at Columbia University; additional summer study at University of Virginia, Georgetown University.

Teacher, elementary schools and secondary schools of Arlington County, secondary schools of New York City, and at Hunter College and Brooklyn College in New York City.

Member: AEA, VEA, NEA, NCTM, Theta Mu Tau, Pi Mu Epsilon, Phi Beta Kappa.

Activities: Instrumental in organizing an Arlington Group of NCTM; Chairman of Arlington Group, responsible for affiliation, arranging programs, etc.; served on Advisory Board for Washington meeting of NCTM in December 1955. Present position involves providing leadership and active participation in curriculum development, in organizing in-service programs for professional improvement, in providing assistance to teachers of secondary mathematics in all phases of classroom activity. Worked closely with the National Science Foundation in Washington D.C. and on the Scholarship Committee of Arlington County Schools, in developing and establishing the "Arlington Plan" of scholarships to teachers for advanced study.

Annie John Williams

Southeastern Region

Teacher of Mathematics, Durham High School, Durham, North Carolina.

A.B., Greensboro College; M.A., University of North Carolina; additional graduate study at North Carolina State College and Duke University; attended Mathematics Institutes of Duke University (1944, 1946–52), New England (1952), and University of Virginia (1953–54); attended Laboratory Conference for Teachers of Science and Mathematics, Duke University, 1953–54, 1956 (Instructional Staff, 1956).

Teacher of Mathematics, Blackstone College, Blackstone, Virginia; Hoke High School, Raeford, North Carolina; Massey Hill High School, Fayetteville, North Carolina; Alexander Graham Junior High School, Fayetteville, North Carolina; Julian S. Carr Junior High School, Durham, North Carolina.

Member: NCTM; NEA; MAA; CASMT; North Carolina Education Association (President, Department of Mathematics, North Central District, 1949–50; State President, Department of Mathematics, 1954–55; Chairman, Program Committee, Mathematics Conference, University of North Carolina, 1954–56); Durham City Education Association (Treasurer, 1945–46); AAUW (Vice-President, Local Branch, 1946–48); Steward, Duke Memorial Methodist Church; Delta Kappa Gamma (Recording Secretary of Eta Chapter, 1956–).

Activities in NCTM: North Carolina State Representative, 1947–54; Delegate to Delegate Assembly, 1953, 1955–56; Member, Agenda Planning Committee for the Seventh Delegate Assembly; Member, Committee on Relations with the NEA; appearance on convention programs.

Publications: "Organizing a Mathematics Club," THE MATHEMATICS TEACHER, February, 1956.

Lottchen Lipp Hunter

Southwestern Region

Curriculum Consultant for the public schools of Wichita and educational institutes in Kansas, and Head of Mathematics Department, Wichita High School West.

A.B. degree, magna cum laude, University of Wichita, M. A., Teachers College, Columbia University, Ph.D. Columbia University.

Teacher, junior and senior high schools of Wichita, Kansas; Summer sessions, University of New Mexico and University of Wichita.

Member: NCTM; Phi Lambda Theta; Kappa Delta Pi; American Association of University Women; National Society for the Study of Education; ASCD; NEA; KSTA; and the Wichita and Kansas Affiliated Groups of NTCM.

Activities: State President of the Kansas Association of Teachers of Mathematics and chairman for the local Affiliated Group.

Publications: Articles in the *Twenty-Second Yearbook* of the NCTM and *The Bulletin* of the Kansas Association of Teachers of Mathematics; curriculum pamphlets: *A Look around Wichita at the Use of Mathematics in the Home, Business, and Industry; Group Dynamics in Secondary School Mathematics; Understanding the Why and How of Mathematics through Things to Make and Do.*

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issue of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING

March 28–30, 1957
Bellevue-Stratford Hotel, Philadelphia, Pennsylvania
M. Albert Linton, William Penn Charter School, Philadelphia, Pennsylvania

JOINT MEETING WITH NEA AND NSTDA

July 1, 1957
Philadelphia, Pennsylvania
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

SUMMER MEETING

August 19–21, 1957
Carleton College, Northfield, Minnesota
Margaret Linster, St. Louis Park High School, Minneapolis 16, Minnesota, or Kenneth O. May, Carleton College, Northfield, Minnesota

Other professional dates

Women's Mathematics Club of Chicago and Vicinity

February 9, 1957
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Ruth Woerner, 11715 S. 82nd Court, Palos Park, Illinois

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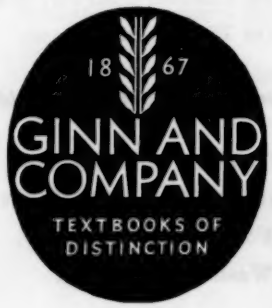

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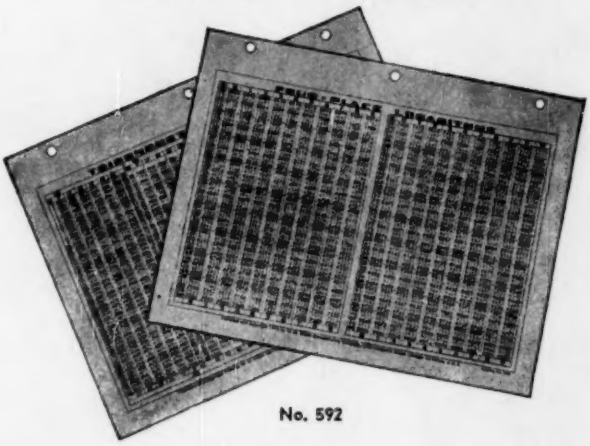

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